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# SETS OF INDEPENDENT POSTULATES FOR THE ARITHMETIC MEAN, THE GEOMETRIC MEAN, THE HARMONIC MEAN, AND THE ROOT-MEAN-SQUARE\*

BY  
EDWARD V. HUNTINGTON

## INTRODUCTION

The four types of means, or averages, considered in this paper are the following: the arithmetic mean ( $A$ ); the geometric mean ( $G$ ); the harmonic mean ( $H$ ); and the root-mean-square ( $S$ ); the familiar definitions being as follows:

$$\begin{aligned} A &= \frac{1}{n} (x_1 + x_2 + \cdots + x_n), \\ G &= (x_1 x_2 \cdots x_n)^{1/n}, \\ H &= \frac{1}{\frac{1}{n} \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)}, \\ S &= \left( \frac{1}{n} (x_1^2 + x_2^2 + \cdots + x_n^2) \right)^{1/2}. \end{aligned}$$

The first three are the classical means, known to the Greeks, while the root-mean-square is of more modern origin. Of the four types, perhaps the most important are the *arithmetic mean* and the *root-mean-square*. Both of these averages are in constant use in mechanics (as in the definitions of center of gravity and radius of gyration), and in the modern theory of statistics (as in the theory of probability, the theory of least squares, and the definition of standard deviation). The *geometric mean* is important chiefly in the construction of index numbers. The *harmonic mean* is little used, except in special investigations.

Each of the four quantities  $A, G, H, S$  is a particular type of the general function of  $n$  variables:  $f(x_1, x_2, \cdots, x_n)$ ; and the purpose of the present

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EDITOR'S NOTE. The typography in this paper and succeeding papers has been altered in various respects from the form originally proposed by the authors, in an effort to adapt mathematical composition to the monotype machine.

paper is to exhibit a number of *sets of independent postulates* by which each of these four types may be distinguished.

Unless otherwise stated, the variables  $x_1, x_2, \dots, x_n$  are supposed to be positive real numbers.

The only similar set of postulates for any of these means, as far as is known to the present writer, is a set of postulates for the arithmetic mean given by R. Schimmack\* in 1909; the "complete independence" of Schimmack's postulates was established by R. D. Beetle† in 1915.

#### PROPERTIES COMMON TO ALL FOUR MEANS

It will be convenient to begin by stating the following general postulates, I-V, which are satisfied by all four of the types of mean here considered. Each of these postulates is a condition imposed upon the as yet undetermined function  $f(x_1, x_2, \dots, x_n)$  of the  $n$  positive real numbers  $x_1, x_2, \dots, x_n$ . Various selections from these postulates will be made below.

$$\text{I. } f(x_1, x_2, \dots, x_i, x_j, \dots, x_n) = f(x_1, x_2, \dots, x_j, x_i, \dots, x_n).$$

That is, the function " $f$ " is independent of the order in which the  $n$  quantities  $x_1, x_2, \dots, x_n$  are taken.

$$\text{II. } f(x_1, x_2, x_3, \dots, x_n) = f(m, m, x_3, \dots, x_n) \text{ where } m = f(x_1, x_2).$$

That is, in computing the " $f$ " of  $n$  quantities, we may replace the first pair,  $x_1, x_2$ , by the " $f$ " of that pair, entered twice.

$$\text{III. } f(kx_1, kx_2, \dots, kx_n) = kf(x_1, x_2, \dots, x_n) \quad (k \text{ positive}).$$

That is, multiplying each of the  $n$  quantities by a positive factor  $k$  has the effect of multiplying the " $f$ " of those quantities by the same factor  $k$ . In other words, the function " $f$ " is independent of the scale in which the quantities  $x_1, x_2, \dots, x_n$  are measured.

$$\text{IV. } f(a, a, \dots, a) = a.$$

That is, if the  $n$  quantities are all equal, then their " $f$ " is equal to their common value.

$$\text{V. } f(x_1, x_2, \dots, x_n) \text{ is positive when all the } x\text{'s are positive.}$$

The postulates III and IV may sometimes be replaced, as we shall see, by the following weaker forms:

\*R. Schimmack, *Der Satz vom arithmetischen Mittel in axiomatischer Begründung*, Mathematische Annalen, vol. 68 (1909), pp. 125-132, and p. 304.

†R. D. Beetle, *On the complete independence of Schimmack's postulates for the arithmetic mean*, Mathematische Annalen, vol. 76 (1915), pp. 444-446.

$$\text{III}'. \quad f(kx_1, kx_2) = kf(x_1, x_2) \quad (k \text{ positive}).$$

$$\text{IV}'. \quad f(1, 1, \dots, 1) = 1.$$

It may be noted that Postulates I, II, III, and III' have the form of "functional equations," in which the function " $f$ " appears on both sides of the equality sign; while Postulates IV, V, and IV' are analogous to the "boundary conditions" of a problem in differential equations, since they tell us something about the actual value of the function in certain cases.

We now turn to properties which are peculiar to the several types of mean. (It will be observed that in each of the following four groups, the first postulate is a "boundary condition," while the other three postulates are "functional equations.") Various selections from these postulates will be made below.

#### POSTULATES PECULIAR TO THE ARITHMETIC MEAN (POSITIVE QUANTITIES\*)

$$A1. \quad f(a, b) = \frac{1}{2}(a + b).$$

$$A2. \quad f(1 - a, 1 - b) = 1 - f(a, b) \quad (a < 1, b < 1).$$

$$A3. \quad f(1 - x_1, 1 - x_2, \dots, 1 - x_n) = 1 - f(x_1, x_2, \dots, x_n) \quad (x_i < 1).$$

$$A4. \quad f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n) \text{ for all values of } A \text{ for which } A - x_i > 0.$$

#### POSTULATES PECULIAR TO THE GEOMETRIC MEAN

$$G1. \quad f(a, b) = (ab)^{1/2}.$$

$$G2. \quad f\left(\frac{1}{a}, \frac{1}{b}\right) = \frac{1}{f(a, b)}.$$

$$G3. \quad f\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = \frac{1}{f(x_1, x_2, \dots, x_n)}.$$

$$G4. \quad f\left(\frac{A}{x_1}, \frac{A}{x_2}, \dots, \frac{A}{x_n}\right) = \frac{A}{f(x_1, x_2, \dots, x_n)},$$

where  $A$  is positive. Here  $(ab)^{1/2}$  means  $+(ab)^{1/2}$  not  $-(ab)^{1/2}$ .

#### POSTULATES PECULIAR TO THE HARMONIC MEAN

$$H1. \quad f(a, b) = \frac{2ab}{a + b}.$$

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\*For further postulates A8, A8', A9, intended for use in the domain of all real or all complex quantities, see Appendix I and Appendix II.

$$H2. \quad \left( \frac{a}{a-1}, \frac{b}{b-1} \right) = \frac{f(a, b)}{f(a, b) - 1}.$$

$$H3. \quad f\left(\frac{x_1}{x_1-1}, \frac{x_2}{x_2-1}, \dots, \frac{x_n}{x_n-1}\right) = \frac{f(x_1, x_2, \dots, x_n)}{f(x_1, x_2, \dots, x_n) - 1} \quad (x_i > 1).$$

$$H4. \quad f\left(\frac{x_1}{Ax_1-1}, \frac{x_2}{Ax_2-1}, \dots, \frac{x_n}{Ax_n-1}\right) = \frac{f(x_1, x_2, \dots, x_n)}{Af(x_1, x_2, \dots, x_n) - 1}$$

for all values of  $A$  for which  $Ax_i - 1 > 0$ .

#### POSTULATES PECULIAR TO THE ROOT-MEAN-SQUARE

$$S1. \quad f(a, b) = \{(x_1^2 + x_2^2)/2\}^{1/2}.$$

$$S2. \quad f\{(1 - a^2)^{1/2}, (1 - b^2)^{1/2}\} = \{1 - [f(a, b)]^2\}^{1/2} \quad (a < 1, b < 1).$$

$$S3. \quad f\{(1 - x_1^2)^{1/2}, (1 - x_2^2)^{1/2}, \dots, (1 - x_n^2)^{1/2}\} \\ = \{1 - [f(x_1, x_2, \dots, x_n)]^2\}^{1/2} \quad (x_i < 1).$$

$$S4. \quad f\{f(A - x_1^2)^{1/2}, (A - x_2^2)^{1/2}, \dots, (A - x_n^2)^{1/2}\} \\ = \{A - [f(x_1, x_2, \dots, x_n)]^2\}^{1/2},$$

for all values of  $A$  for which  $A - x_i^2 > 0$ . Here again,  $x^{1/2}$  means  $+x^{1/2}$  not  $-x^{1/2}$ .

#### SETS OF INDEPENDENT POSTULATES FOR EACH TYPE OF MEAN

Among these general and special properties there are, of course, many redundancies. The purpose of the present paper is to select, for each type of mean, sets of *independent* postulates; that is, sets of postulates which, while sufficient to determine uniquely the type of mean in question, shall at the same time be free from all redundancies. Such sets can be selected in a variety of ways, as in the following tables. These tables give the postulates belonging to each set, and also the list of examples which will later be used to prove the independence of the postulates in that set.

It will be observed that in each group the fifth and sixth sets are obtained from the second set by replacing either III or IV by the weaker form III' or IV'; while the seventh set is obtained from the first by replacing IV by IV' and then adding III. No further replacements of III or IV by III' or IV' are possible, as we shall see by Examples *A* III 7, *G* III 7, *H* III 7, and *S* III 7.

Thus, a set of postulates comprising *A*1, *A*2, *A*3, I, II, III', IV', V would not be sufficient to determine the arithmetic mean; *G*1, *G*2, *G*3, I, II, III', IV', V would not be sufficient for the geometric mean; nor *H*1, *H*2, *H*3, I, II, III', IV', V for the harmonic mean; nor *S*1, *S*2, *S*3, I, II, III', IV', V for the root-mean-square.

## For the Arithmetic Mean (positive quantities\*)

Set	Postulates					Examples Used				
A1	A1	I	II		IV	0	A I	A II		A IV
A2	A2	I	II	III	IV	0	A I	A II	A III	A IV
A3	A3	I	II	III		0	A I	A II	A III	
A4	A4	I	II			0	A I	A II		
A5	A2	I	II	III'	IV	0	A I	A II	A III	A IV
A6	A2	I	II	III	IV'	0	A I	A II	A III	A IV
A7	A1	I	II	III	IV'	0	A I	A II	A III 7	A IV

## For the Geometric Mean

Set	Postulates					Examples Used				
G1	G1	I	II		IV	0	G I	G II		G IV
G2	G2	I	II	III	IV	0	G I	G II	G III	G IV
G3	G3	I	II	III		0	G I	G II	G III	
G4	G4	I	II			0	G I	G II		
G5	G2	I	II	III'	IV	0	G I	G II	G III	G IV
G6	G2	I	II	III	IV'	0	G I	G II	G III	G IV
G7	G1	I	II	III	IV'	0	G I	G II	G III 7	G IV

## For the Harmonic Mean

Set	Postulates					Examples Used				
H1	H1	I	II		IV	0	H I	H II		H IV
H2	H2	I	II	III	IV	0	H I	H II	H III	H IV
H3	H3	I	II	III		0	H I	H II	H III	
H4	H4	I	II			0	H I	H II		
H5	H2	I	II	III'	IV	0	H I	H II	H III	H IV
H6	H2	I	II	III	IV'	0	H I	H II	H III	H IV
H7	H1	I	II	III	IV'	0	H I	H II	H III 7	H IV

\*For further Sets A6'', A7'', A8, A9, intended for use in the real or complex domain, see Appendices I and II.

## For the Root-Mean-Square

Set	Postulates					Examples Used				
S1	S1	I	II		IV	0	SI	S II		S IV
S2	S2	I	II	III	IV	0	SI	S II	S III	S IV
S3	S3	I	II	III		0	SI	S II	S III	
S4	S4	I	II			0	SI	S II		
S5	S2	I	II	III'	IV	0	SI	S II	S III	S IV
S6	S2	I	II	III	IV'	0	SI	S II	S III	S IV
S7	S1	I	II	III	IV'	0	SI	S II	S III 7	S IV

## EXAMPLES USED IN PROOFS OF INDEPENDENCE

To establish the independence of the postulates of each set, we exhibit, in the usual way, a list of examples of functions  $f(x_1, x_2, \dots, x_n)$  which satisfy some but not all of the postulates. In the following table the postulates satisfied by each example are stated explicitly, opposite the number of that example. (A dash, —, indicates that the postulate is not satisfied.) The list of examples is given immediately below the table.\*

As an illustration of the use of these examples, consider Postulate III in Set G3. Example G III satisfies Postulates G3, I, II, and V, but fails on Postulate III. Hence III is not a consequence of G3, I, II, V; that is, Postulate III is not a redundancy in Set G3. Similarly for each of the other postulates in this set and in each of the other sets.

Ex. 0. 
$$f() = \{(x_1^3 + x_2^3 + \dots + x_n^3)/n\}^{1/3}.$$

This example satisfies all the general postulates I–V, but none of the special postulates A1–A4, G1–G4, H1–H4, S1–S4. To see that II is satisfied, note that

$$m = f(x_1, x_2) = \{(x_1^3 + x_2^3)/2\}^{1/3},$$

so that  $m^3 + m^3 = x_1^3 + x_2^3$ .

Ex. AI. 
$$f() = \frac{x_1 + x_2 + 3x_3 + 4x_4 + \dots + nx_n}{1 + 1 + 3 + 4 + \dots + n}.$$

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\*For further Examples: 0', AII', A III'', A IV', A IV'', AII'', A IX, and AX, for the arithmetic mean in the real or complex domain, see Appendices I and II.

	A1	A2	A3	A4	I	II	III	IV	III'	IV'	V
Ex. 0	—	—	—	—	I	II	III	IV	III'	IV'	V
Ex. A I	A1	A2	A3	A4	—	II	III	IV	III'	IV'	V
Ex. A II	A1	A2	A3	A4	I	—	III	IV	III'	IV'	V
Ex. A III	—	A2	A3	—	I	II	—	IV	—	IV'	V
Ex. A IV	A1	A2	—	—	I	II	III	—	III'	—	V
Ex. A III 7	A1	A2	A3	—	I	II	—	—	III'	IV'	V

	G1	G2	G3	G4	I	II	III	IV	III'	IV'	V
Ex. 0	—	—	—	—	I	II	III	IV	III'	IV'	V
Ex. G I	G1	G2	G3	G4	—	II	III	IV	III'	IV'	V
Ex. G II	G1	G2	G3	G4	I	—	III	IV	III'	IV'	V
Ex. G III	—	G2	G3	—	I	II	—	IV	—	IV'	V
Ex. G IV	G1	G2	—	—	I	II	III	—	III'	—	V
Ex. G III 7	G1	G2	G3	—	I	II	—	—	III'	IV'	V
Ex. G V	—	G2	G3	G4	I	II	III	IV	III'	IV'	—

	H1	H2	H3	H4	I	II	III	IV	III'	IV'	V
Ex. 0	—	—	—	—	I	II	III	IV	III'	IV'	V
Ex. H I	H1	H2	H3	H4	—	II	III	IV	III'	IV'	V
Ex. H II	H1	H2	H3	H4	I	—	III	IV	III'	IV'	V
Ex. H III	—	H2	H3	—	I	II	—	IV	—	IV'	V
Ex. H IV	H1	H2	—	—	I	II	III	—	III'	—	V
Ex. H III 7	H1	H2	H3	—	I	II	—	—	III'	IV'	V
Ex. H V	—	H2	H3	H4	I	II	III	IV	III'	IV'	—

	S1	S2	S3	S4	I	II	III	IV	III'	IV'	V
Ex. 0	—	—	—	—	I	II	III	IV	III'	IV'	V
Ex. S I	S1	S2	S3	S4	—	II	III	IV	III'	IV'	V
Ex. S II	S1	S2	S3	S4	I	—	III	IV	III'	IV'	V
Ex. S III	—	S2	S3	—	I	II	—	IV	—	IV'	V
Ex. S IV	S1	S2	—	—	I	II	III	—	III'	—	V
Ex. S III 7	S1	S2	S3	—	I	II	—	—	III'	IV'	V

Ex. A II.  $f()$  = the "median" of the quantities  $x_1, x_2, \dots, x_n$ , if  $n$  is odd, or the arithmetic mean of the "median-pair" if  $n$  is even.\* To see that II fails note that  $f(5, 3, 7) = 5$ , while  $f(4, 4, 7) = 4$ .

Ex. A III.  $f(x, x, \dots, x) = x$  when all the  $x$ 's are equal; otherwise  $f() = \frac{1}{2}$ . This example fails on III and III', and on A1 and A4; but it satisfies A2 and A3.

$$\text{Ex. A IV. } f() = (x_1 + x_2 + \dots + x_n)/2.$$

This example fails on IV and IV', and on A3 and A4; but it satisfies A1 and A2.

Ex. A III 7.  $f(x_1, x_2) = (x_1 + x_2)/2$ ; but when  $n > 2$ ,  $f() = 1$  or  $\frac{1}{2}$ , according as  $(x_1 + x_2 + \dots + x_n)/n$  is equal to 1 or not equal to 1. This example is used only in Set A7, to prove the independence of Postulate III in that set. It fails on III and IV, but satisfies III' and IV'. To see that it satisfies A2 and A3, note that if all the  $x$ 's are less than 1, the value of  $(x_1 + x_2 + \dots + x_n)/n$  must be less than 1, and hence  $f() = \frac{1}{2}$ .

Ex. G I.  $f() = (x_1 x_2 x_3^2 x_4^4 \dots x_n^n)^{1/p}$ , where  $p = 1 + 1 + 3 + 4 + \dots + n$ .

Ex. G II.  $f()$  = the median of the  $n$  quantities if  $n$  is odd, or the geometric mean of the median-pair if  $n$  is even. To see that II fails, note that  $f(4, 9, 5) = 5$ , while  $f(6, 6, 5) = 6$ .

Ex. G III.  $f(x, x, \dots, x) = x$  when all the  $x$ 's are equal; otherwise  $f() = 1$ .

Ex. G IV.  $f() = \{(x_1 x_2 \dots x_n)/(n-1)\}^{1/n}$ . To see that II holds, note that  $m = f(x_1, x_2) = (x_1 x_2)^{1/2}$ , so that  $mm = x_1 x_2$ .

Ex. G III 7.  $f(x_1, x_2) = (x_1 x_2)^{1/2}$ ; but when  $n > 2$ ,  $f() = 1$ . This example is used only in Set G7, to prove the independence of Postulate III in that set.

Ex. G V.  $f(x, x, \dots, x) = x$  when all the  $x$ 's are equal; otherwise  $f() = -|x_1 x_2 \dots x_n|^{1/n}$ , where the expression of which the  $n$ th root is taken is the absolute value of the product of the  $x$ 's without regard to sign.

$$\text{Ex. H I. } f() = \frac{1 + 1 + 3 + 4 + \dots + n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{3}{x_3} + \frac{4}{x_4} + \dots + \frac{n}{x_n}}.$$

Here when  $n = 2$ ,

$$f(x_1, x_2) = \frac{2x_1 x_2}{x_1 + x_2}.$$

Ex. H II.  $f()$  = the median of the  $n$  quantities if  $n$  is odd, or the harmonic mean of the median-pair if  $n$  is even. To see that II fails, note that  $f(3, 6, 2) = 3$ , while  $f(4, 4, 2) = 4$ .

\*Here the median (or median-pair) of  $n$  positive quantities is defined by arranging the  $n$  quantities in a series in order of magnitude, and picking out the middle item (or mid-pair) in this series.

Ex. *H III*.  $f(x, x, \dots, x) = x$  if all the  $x$ 's are equal; otherwise  $f() = 2$ .

Ex. *H IV*. 
$$f() = 2 / \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right).$$

Ex. *H III 7*.  $f(x_1, x_2) = 2x_1x_2/(x_1+x_2)$ ; but when  $n > 2$ ,  $f() = 1$  or  $2$ , according as  $(1/n)[(1/x_1) + (1/x_2) + \dots + (1/x_n)]$  is equal to  $1$  or not equal to  $1$ . This example is used only in Set *H7*, to prove the independence of Postulate *III* in that set.

Ex. *H V*.  $f(x, x, \dots, x) = x$  when all the  $x$ 's are equal; otherwise  $f() = 0$ .

Ex. *S I*. 
$$f() = \left( \frac{x_1^2 + x_2^2 + 3x_3^2 + 4x_4^2 + \dots + nx_n^2}{1 + 1 + 3 + 4 + \dots + n} \right)^{1/2}.$$

To see that this example satisfies *II*, note that  $m = f(x_1, x_2) = \{(x_1^2 + x_2^2)/2\}^{1/2}$ , so that  $m^2 + m^2 = x_1^2 + x_2^2$ .

Ex. *S II*.  $f()$  = the median of the  $n$  quantities if  $n$  is odd, or the root-mean-square of the median-pair if  $n$  is even. To see that *II* fails, note that  $f(6, 8, 4) = 6$ , while  $f(50^{1/2}, 50^{1/2}, 4) = 50^{1/2}$ .

Ex. *S III*.  $f(x, x, \dots, x) = x$  when all the  $x$ 's are equal; otherwise  $f() = (1/2)^{1/2}$ .

Ex. *S IV*.  $f() = \{(x_1^2 + x_2^2 + \dots + x_n^2)/2\}^{1/2}$ .

Ex. *S III 7*.  $f(x_1, x_2) = \{(x_1^2 + x_2^2)/2\}^{1/2}$ ; but when  $n > 2$ ,  $f() = 1$  or  $(1/2)^{1/2}$ , according as  $(x_1^2 + x_2^2 + \dots + x_n^2)/n$  is equal to  $1$  or not equal to  $1$ . This example is used only in Set *S7*, to prove the independence of Postulate *III* in that set.

#### PROOFS OF THEOREMS

In the following paragraphs, we give the proof that each of the foregoing sets of postulates is sufficient to define the type of mean in question.

**THEOREM A (A).** *Proof of A from A1, I, II, and IV.*

Let  $q$  be any positive quantity which is less than  $(1/n)$ th of the smallest of the  $x$ 's. Then by *II* and *A1*,

$$f(x_1, x_2, x_3, \dots, x_n) = f(q, [x_1 + x_2 - q], x_3, \dots, x_n),$$

since each side equals  $f(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2), x_3, \dots, x_n)$ , and all the arguments are positive.

By successive applications of this result, in view of *I*, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f(q, q, [x_1 + x_2 + x_3 - 2q], x_4, \dots, x_n) \\ &= f(q, q, q, [x_1 + x_2 + x_3 + x_4 - 3q], x_5, \dots, x_n) \\ &= f(q, q, \dots, [x_1 + x_2 + \dots + x_n - (n-1)q]). \end{aligned}$$

Now take  $a = (x_1 + x_2 + \dots + x_n)/n$ . Then, putting each  $x$  equal to  $a$ ,  $f(a, a, \dots, a) = f(q, q, \dots, [na - (n-1)q]) = a$ , by Postulate IV. But  $x_1 + x_2 + \dots + x_n = na$ , so that

$$f(x_1, x_2, \dots, x_n) = f(q, q, \dots, [na - (n-1)q]).$$

Hence

$$f(x_1, x_2, \dots, x_n) = (x_1 + x_2 + \dots + x_n)/n.$$

Hence any function  $f$  which satisfies the postulates of Set A1 must be identical with the arithmetic mean,  $A$ .

THEOREM A (B). *Proof of A1 from A2, I, and III or III'.*

$$\begin{aligned} f(a, b) &= (a+b)f\left(\frac{a}{a+b}, \frac{b}{a+b}\right), \text{ by III or III';} \\ &= (a+b)f\left(1 - \frac{b}{a+b}, 1 - \frac{a}{a+b}\right) \\ &= (a+b)\left[1 - f\left(\frac{b}{a+b}, \frac{a}{a+b}\right)\right], \text{ by A2;} \\ &= (a+b) - f(b, a), \text{ by III or III';} \\ &= (a+b) - f(a, b), \text{ by I.} \end{aligned}$$

Hence  $f(a, b) = (a+b)/2$ , which is A1.

This proof shows that any function which satisfies the postulates of Set A2 or Set A5 will also satisfy the postulates of Set A1, and hence be identical with the arithmetic mean.

THEOREM A (C). *Proof of IV from A3, III.*

From A3, putting  $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, \dots, x_n = \frac{1}{2}$ , we have

$$f(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) = 1 - f(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}),$$

whence  $f(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) = \frac{1}{2}$ . Hence by III,  $f(a, a, \dots, a) = a$ , which is IV.

This proof shows (since Postulate A2 follows at once from Postulate A3) that any function which satisfies Set A3 will also satisfy Set A2.

THEOREM A (D). *Proof of A1 and IV from A4, I.*

From A4, putting  $A = a+b, x_1 = a, x_2 = b$ , and  $n=2$ , we have

$$f(a+b-a, a+b-b) = a+b-f(a, b);$$

whence by I,  $f(a, b) = a+b-f(a, b)$ . Therefore  $f(a, b) = (a+b)/2$ , which is A1.



From G3, putting  $x_1=1, x_2=1, \dots, x_n=1$ , we have  $f(1, 1, \dots, 1) = 1/f(1, 1, \dots, 1)$ , whence  $f(1, 1, \dots, 1) = 1$  or  $-1$ . Hence by V,  $f(1, 1, \dots, 1) = 1$ . Then by III,  $f(a, a, \dots, a) = a$ , which is IV.

This proof shows (since G2 follows at once from G3) that Set G3 reduces to Set G2, and hence determines the geometric mean.

THEOREM G (D). *Proof of G1 and IV from G4, I, V.*

From G4, putting  $a=ab, x_1=a, x_2=b$ , and  $n=2$ , we have

$$f\left(\frac{ab}{a}, \frac{ab}{b}\right) = \frac{ab}{f(a, b)}.$$

Hence by I,

$$f(a, b) = \frac{ab}{f(a, b)}.$$

Therefore  $f(a, b) = (ab)^{1/2}$  or  $-(ab)^{1/2}$ . But the negative value is excluded, by V. Hence  $f(a, b) = (ab)^{1/2}$ , which is G1.

Again, from G4, putting  $x$  equal to  $a$ , and  $A=a^2$ , we have

$$f(a, a, \dots, a) = \frac{a^2}{f(a, a, \dots, a)},$$

whence, by V,  $f(a, a, \dots, a) = a$ , which is IV.

This proof shows that Set G4 reduces to Set G1, and hence determines the geometric mean.

THEOREM G (E). Since IV follows at once from III and IV', we see that Set G6 reduces to Set G2, and that Set G7 reduces to Set G1; hence Set G6 and Set G7 determine the geometric mean.

THEOREM H (A). *Proof of H from H1, I, II, IV.*

Let  $q$  be any positive quantity which is greater than  $n$  times the largest of the  $x$ 's. Then by II and H1,

$$f(x_1, x_2, \dots, x_n) = f\left(q, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{q}}, x_3, \dots, x_n\right),$$

since each side is equal to

$$f\left(\frac{2x_1x_2}{x_1 + x_2}, \frac{2x_1x_2}{x_1 + x_2}, x_3, \dots, x_n\right),$$

and all the arguments are positive. By successive applications of this result, using I, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f\left(q, q, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{2}{q}}, x_4, \dots, x_n\right) \\ &\quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ &= f\left(q, q, \dots, \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - \frac{n-1}{q}}\right). \end{aligned}$$

Now take

$$a = \left( \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} \right).$$

Then putting each  $x$  equal to  $a$ ,

$$f(a, a, \dots, a) = \left( q, q, \dots, \frac{1}{\frac{n}{a} - \frac{n-1}{q}} \right) = a,$$

by Postulate IV. But

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{n}{a},$$

so that

$$f(x_1, x_2, \dots, x_n) = f\left(q, q, \dots, \frac{1}{\frac{n}{a} - \frac{n-1}{q}}\right) = a.$$

Hence

$$f(x_1, x_2, \dots, x_n) = \frac{1}{\frac{1}{n} \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}.$$

This proof shows that Set  $H1$  determines the harmonic mean.

THEOREM H (B). *Proof of H1 from H2, I, III or III', V.*

$$\begin{aligned}
 f(a, b) &= \frac{ab}{a+b} f\left(\frac{a+b}{b}, \frac{a+b}{a}\right), \text{ by III or III' ;} \\
 &= \frac{ab}{a+b} f\left(\frac{\frac{a+b}{a}}{\frac{a+b}{b}-1}, \frac{\frac{a+b}{b}}{\frac{a+b}{a}-1}\right) \\
 &= \frac{ab}{a+b} \frac{f\left(\frac{a+b}{a}, \frac{a+b}{b}\right)}{f\left(\frac{a+b}{a}, \frac{a+b}{b}\right) - 1}, \text{ by H2.}
 \end{aligned}$$

Now by V,  $f(a, b)$  is not zero. Hence

$$\begin{aligned}
 \frac{1}{f(a, b)} &= \frac{1}{\frac{ab}{a+b}} - \frac{1}{\frac{ab}{a+b} f\left(\frac{a+b}{a}, \frac{a+b}{b}\right)} \\
 &= \frac{a+b}{ab} - \frac{1}{f(b, a)}, \text{ by III or III' ;} \\
 &= \frac{a+b}{ab} - \frac{1}{f(a, b)}, \text{ by I.}
 \end{aligned}$$

Hence

$$f(a, b) = \frac{2ab}{a+b},$$

which is H1.

This proof shows that Set H2 and Set H5 reduce to Set H1, and hence determine the harmonic mean.

THEOREM H (c). *Proof of IV from H3, III, V.*

From H3, putting  $x_1=2, x_2=2, \dots, x_n=2$ , we have

$$f(2, 2, \dots, 2) = \frac{f(2, 2, \dots, 2)}{f(2, 2, \dots, 2) - 1}.$$

Hence  $f(2, 2, \dots, 2)[f(2, 2, \dots, 2) - 2] = 0$ . But by V,  $f(2, 2, \dots, 2)$  is not zero. Hence  $f(2, 2, \dots, 2) = 2$ . Hence by III,  $f(a, a, \dots, a) = a$ , which is IV.

This proof shows (since *H2* follows at once from *H3*) that Set *H3* reduces to Set *H2*, and hence determines the harmonic mean.

THEOREM H (D). *Proof of H1 and IV from H4.*

From *H4*, putting  $A = (a+b)/(ab)$ ,  $x_1 = a$ ,  $x_2 = a$ , and  $n=2$ , we have

$$f(b, a) = \frac{(ab)f(a, b)}{(a+b)f(a, b) - ab},$$

whence by I,

$$f(a, b) = \frac{(ab)f(a, b)}{(a+b)f(a, b) - ab},$$

where by V,  $f(a, b)$  is not zero; hence  $f(a, b) = 2ab/(a+b)$ , which is *H1*.

Again, from *H4*, putting each  $x$  equal to  $a$ , and  $A = 2/a$ , we have

$$f(a, a, \dots, a) = \frac{af(a, a, \dots, a)}{2f(a, a, \dots, a) - a},$$

where by V,  $f(a, a, \dots, a)$  is not zero; hence  $f(a, a, \dots, a) = a$ , which is IV.

This proof shows that Set *H4* reduces to Set *H1*, and hence determines the harmonic mean.

THEOREM H (E). Since IV follows at once from III and IV', we see that Set *H6* and Set *H7* reduce to Set *H2* and Set *H1* respectively, and hence determine the harmonic mean.

THEOREM S (A). *Proof of S from S1, I, II, IV.*

Let  $q$  be any positive quantity which is smaller than  $(1/n)$ th of the smallest of the  $x$ 's. Then by II and S1,

$$f(x_1, x_2, x_3, \dots, x_n) = f\{q, (x_1^2 + x_2^2 - q^2)^{1/2}, x_3, \dots, x_n\},$$

since each side is equal to  $f\{[(x_1^2 + x_2^2)/2]^{1/2}, [(x_1^2 + x_2^2)/2]^{1/2}, x_3, \dots, x_n\}$ , and all the arguments are positive. By successive applications of this result, in view of I, we have

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f\{q, q, (x_1^2 + x_2^2 + x_3^2 - 2q^2)^{1/2}, x_4, \dots, x_n\} \\ &\dots \dots \dots \\ &= f\{q, q, \dots, (x_1^2 + x_2^2 + \dots + x_n^2 - (n-1)q^2)^{1/2}\}. \end{aligned}$$

Now take  $a = \{(x_1^2 + x_2^2 + \dots + x_n^2)/n\}^{1/2}$ . Then putting each  $x$  equal to  $a$ ,  $f(a, a, \dots, a) = f\{q, q, \dots, (na^2 - (n-1)q^2)^{1/2}\} = a$ , by IV.

But  $x_1^2 + x_2^2 + \dots + x_n^2 = na^2$ , so that

$$f(x_1, x_2, \dots, x_n) = f\{q, q, \dots, (na^2 - (n-1)q^2)^{1/2}\} = a.$$

Hence 
$$f(x_1, x_2, \dots, x_n) = \left( \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} \right)^{1/2}.$$

This proof shows that Set S1 determines the root-mean-square.

THEOREM S (B). *Proof of S1 from S2, I, III or III'.*

$$\begin{aligned} f(a, b) &= (a^2 + b^2)^{1/2} f\left(\frac{a}{(a^2 + b^2)^{1/2}}, \frac{b}{(a^2 + b^2)^{1/2}}\right), \text{ by III or III';} \\ &= (a^2 + b^2)^{1/2} f\left[\left(1 - \frac{b^2}{a^2 + b^2}\right)^{1/2}, \left(1 - \frac{a^2}{a^2 + b^2}\right)^{1/2}\right] \\ &= (a^2 + b^2)^{1/2} \left(1 - \left[f\left(\frac{b}{(a^2 + b^2)^{1/2}}, \frac{a}{(a^2 + b^2)^{1/2}}\right)\right]^2\right)^{1/2}, \text{ by S2;} \\ &= \{(a^2 + b^2) - [f(b, a)]^2\}^{1/2}, \text{ by III or III';} \\ &= \{(a^2 + b^2) - [f(a, b)]^2\}^{1/2} \text{ by I.} \end{aligned}$$

Hence  $2[f(a, b)]^2 = a^2 + b^2$ , so that

$$f(a, b) = \{(a^2 + b^2)/2\}^{1/2} \text{ or } -\{(a^2 + b^2)/2\}^{1/2}.$$

It remains to exclude the negative value (without using V). By S2,  $f(a, b)$  is positive whenever  $a < 1$  and  $b < 1$ ; hence by III,  $f(a, b)$  is positive for all positive values of  $a$  and  $b$ . Hence  $f(a, b) = \{(a^2 + b^2)/2\}^{1/2}$ .

This proof shows that Set S2 and Set S5 reduce to Set S1, and hence determine the root-mean-square.

THEOREM S (C). *Proof of IV from S3, III.*

From S3, putting  $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, \dots, x_n = \frac{1}{2}$ , we have

$$\begin{aligned} f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\} \\ = (1 - [f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\}]^2)^{1/2} \end{aligned}$$

which by definition of the square root sign, is not negative. Hence

$$f\{(1/2)^{1/2}, (1/2)^{1/2}, \dots, (1/2)^{1/2}\} = (1/2)^{1/2}.$$

Hence by III,

$$f(a, a, \dots, a) = a.$$

This proof shows (since S2 follows at once from S3) that Set S3 reduces to Set S2, and hence determines the root-mean-square.

THEOREM S (D). *Proof of S1 and IV from S4, I.*

From S4, putting  $A = a^2 + b^2, x_1 = a, x_2 = b$ , and  $n = 2$ , we have

$$f(b, a) = (a^2 + b^2 - [f(a, b)]^2)^{1/2}.$$

Hence, by I,  $f(a, b) = (a^2 + b^2 - [f(a, b)]^2)^{1/2}$ , which is not negative. Hence  $f(a, b) = \{(a^2 + b^2)/2\}^{1/2}$ .

Again, from *S4*, putting  $x$  equal to  $a$ , and  $A = 2a^2$ , we have

$$f(a, a, \dots, a) = \{2a^2 - [f(a, a, \dots, a)]^2\}^{1/2},$$

which is not negative; hence  $f(a, a, \dots, a) = a$ , which is Postulate IV.

This proof shows that Set *S4* reduces to Set *S1*, and hence determines the root-mean-square.

THEOREM S (E). Since IV follows at once from III and IV', we see that Set *S6* and Set *S7* reduce to Set *S2* and Set *S1* respectively, and hence determine the root-mean-square.

#### APPENDIX I. THE ARITHMETIC MEAN IN THE DOMAIN OF ALL REAL NUMBERS

In all the preceding sets of postulates, the  $x$ 's in the function  $f(x_1, x_2, \dots, x_n)$  have been assumed to be *positive real quantities*. In the case of the geometric mean, the harmonic mean, and the root-mean-square, this restriction is a customary one, in order to insure that the function shall always be finite and single-valued. In the case of the arithmetic mean, however, the restriction is not essential.

This appendix, therefore, is devoted to a consideration of sets of postulates for the arithmetic mean in the domain of all real numbers.

For this purpose, we introduce, in addition to the postulates I-V, III', IV', A1-A4, given above, the following postulates:

POSTULATE IV''.  $f(-1, -1, \dots, -1) = -1$ .

POSTULATE A8.  $f(A+x_1, A+x_2, \dots, A+x_n) = A + f(x_1, x_2, \dots, x_n)$ .

POSTULATE A8'.  $f(-x_1, -x_2, \dots, -x_n) = -f(x_1, x_2, \dots, x_n)$ .

The results obtained may be summarized as follows:

In the first place, Sets A1, A2, A3, A4, A5, given above for the case of positive reals, are valid just as they stand for the case of all reals. (The necessary modifications in the proofs are given below.)

In the second place, Sets A6 and A7 are not valid for the case of all reals (see Example A IV'' below); but they can be made so by the addition of Postulate IV''.

In the third place, Postulates I, II, A8, A8' form a set (due essentially to Schimmack\*), which is valid for the case of all reals, but cannot be used in the case of positive reals; this we shall call Set A8.

\*Instead of our Postulate II, Schimmack (loc. cit.) uses the following postulate:

$f(x_1, x_2, \dots, x_{n-1}, x_n) = f(m, m, \dots, m, x_n)$ , where  $m = f(x_1, x_2, \dots, x_{n-1})$ ; and instead of our Example A1, the following example:

$$f() = (x_1/2^{n-1}) + (x_2/2^{n-1}) + (x_3/2^{n-2}) + (x_4/2^{n-3}) + \dots + (x_{n-1}/2^2) + (x_n/2).$$

The complete list of sets of independent postulates for the arithmetic mean, for the case of all reals, is then as follows:

Set	Postulates				Examples Used			
A1	A1	I	II	IV	0	A I	A II'	A IV
A2	A2	I	II	III IV	0	A I	A II'	A III A IV
A3	A3	I	II	III	0	A I	A II'	A III
A4	A4	I	II		0	A I	A II'	
A5	A2	I	II	III' IV	0	A I	A II'	A III A IV
A6''	A2	I	II	III IV' IV''	0	A I	A II'	A III'' A IV' A IV''
A7''	A1	I	II	III IV' IV''	0	A I	A II'	A III'' A IV' A IV''
A8	A8	I	II	A8'	0	A I	A II'	0'

The new examples in the independence proofs are the following:

Ex. A II'. The same as Ex. A II, for all reals, understanding "the order of magnitude" in the algebraic sense.

Ex. A III''. When  $n=2$ ,  $f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$ ; when  $n>2$ ,

$$f(x_1, x_2, \dots, x_n) = 1 \text{ or } -1$$

according as  $(x_1+x_2+\dots+x_n)/n$  is equal to 1 or not equal to 1.

Ex. A IV'. When  $n=2$ ,  $f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$ ; when  $n>2$ ,

$$f() = -|(x_1+x_2+\dots+x_n)/n|,$$

where the vertical bars mean "the absolute value of."

Ex. A IV''. When  $n=2$ ,  $f(x_1, x_2) = \frac{1}{2}(x_1+x_2)$ ; when  $n>2$ ,

$$f() = |(x_1+x_2+\dots+x_n)/n|.$$

Ex. 0'.  $f()$  = the maximum (in the algebraic sense) of the  $n$  real quantities  $x_1, x_2, \dots, x_n$ .

The following table shows the properties of all the examples used in the case of the real domain.

	A1	A2	A3	A4	A8	A8'	I	II	III	IV	III'	IV'	IV''
Ex. 0	—	—	—	—	—	A8'	I	II	III	IV	III'	IV'	IV''
Ex. 0'	—	—	—	—	A8	—	I	II	III	IV	III'	IV'	IV''
Ex. A I	A1	A2	A3	A4	A8	A8'	—	II	III	IV	III'	IV'	IV''
Ex. A II'	A1	A2	A3	A4	A8	A8'	I	—	III	IV	III'	IV'	IV''
Ex. A III	—	A2	A3	—	—	—	I	II	—	IV	—	IV'	IV''
Ex. A IV	A1	A2	—	—	—	A8'	I	II	III	—	III'	—	—
Ex. A III''	A1	A2	—	—	—	—	I	II	—	—	III'	IV'	IV''
Ex. A IV'	A1	A2	—	—	—	—	I	II	III	—	III'	—	IV''
Ex. A IV''	A1	A2	—	—	—	—	I	II	III	—	III'	IV'	—

The proofs of the theorems, adapted to the domain of all reals, are as follows:

THEOREM A (A'). *Proof of A from A1, I, II, IV. (Real domain.)*

Since there is now no necessity for keeping the arguments positive, the proof of Theorem A (A) can be simplified by putting  $q=0$ . Hence Set A1 determines the arithmetic mean.

THEOREM A (B'). *Proof of A1 from A2, I, III or III'. (Real domain.)*

If  $a$  and  $b$  are positive,  $f(a, b) = \frac{1}{2}(a+b)$ , as in the proof of Theorem A (B).

If  $a$  and  $b$  are any real numbers, let  $p = 1/(|a| + |b| + 1)$ ; then  $1 - pa$  and  $1 - pb$  will be positive, so that

$$f(1 - pa, 1 - pb) = \frac{1}{2}(1 - pa + 1 - pb) = 1 - p(a+b)/2.$$

But  $f(1 - pa, 1 - pb) = 1 - f(pa, pb)$ , by A2;  $= 1 - pf(a, b)$ , by III or III'. Therefore  $f(a, b) = (a+b)/2$ , for all real values of  $a$  and  $b$ . Hence Set A2 and Set A5 reduce to Set A1.

THEOREM A (C'). *Proof of IV from A3, I, III. (Real domain.)*

By III,  $f(0, 0, \dots, 0) = 2f(0, 0, \dots, 0)$ ; hence  $f(0, 0, \dots, 0) = 0$ .

From A3,  $f(1, 1, \dots, 1) = 1 - f(0, 0, \dots, 0) = 1$ . Hence by III,

$$f(a, a, \dots, a) = a,$$

whenever  $a$  is positive.

From A3,  $f(2, 2, \dots, 2) = 1 - f(-1, -1, \dots, -1)$ , whence

$$2 = 1 - f(-1, -1, \dots, -1),$$

whence  $f(-1, -1, \dots, -1) = -1$ . Hence by III,

$$f(a, a, \dots, a) = a$$

whenever  $a$  is negative.

Since A2 follows immediately from A3, this proof shows that Set A3 reduces to Set A2.

THEOREM A (D'). *Proof of A1 and IV from A4, I, II. (Real domain.)*

If  $a$  and  $b$  are positive, put  $A = a+b$ ,  $x_1 = a$ ,  $x_2 = b$ , and  $n = 2$ , in A4; then  $f(b, a) = a+b - f(a, b)$ , whence by I,  $f(a, b) = a+b - f(a, b)$ , whence

$$f(a, b) = \frac{1}{2}(a+b),$$

whenever  $a$  and  $b$  are positive.

If  $a$  and  $b$  are any reals, let  $p =$  a large positive quantity, so that  $\bar{p} + a$ ,  $p + b$ , and  $p + a + b$  will certainly be positive. Putting  $A = p + a + b$ ,  $x_1 = a$ ,  $x_2 = b$ , and  $n = 2$ , in A4, we have

$$f(p + b, p + a) = p + a + b - f(a, b),$$

whence

$$(p + b + p + a)/2 = p + a + b - f(a, b),$$

whence  $f(a, b) = (a + b)/2$ , which is Postulate A1 for all real values of  $a$  and  $b$ .

If  $a$  is positive, put  $x_1 = a, x_2 = a, \dots, x_n = a$ , and  $A = 2a$ , in A4; then  $f(a, a, \dots, a) = 2a - f(a, a, \dots, a)$ , whence  $f(a, a, \dots, a) = a$ .

Again, if  $a$  is positive, put  $x_1 = 0, x_2 = 0, \dots, x_n = 0, A = a$ , in A4; then  $f(a, a, \dots, a) = a - f(0, 0, \dots, 0)$ , whence  $f(0, 0, \dots, 0) = 0$ .

If  $a$  is negative, put  $x_1 = a, x_2 = a, \dots, x_n = a$ , and  $A = 0$ , in A4; then  $f(-a, -a, \dots, -a) = -f(a, a, \dots, a)$ , whence, since  $-a$  is positive,  $-a = -f(a, a, \dots, a)$ ; hence  $f(a, a, \dots, a) = a$ .

Therefore  $f(a, a, \dots, a) = a$  for all real values of  $a$ , which is Postulate IV.

This proof shows that Set A4 reduces to Set A1.

**THEOREM A (E').** Since IV follows at once from III, IV', IV'', we see that Set A6'' and Set A7'' reduce to Sets A2 and A1 respectively, in the domain of all reals.

**THEOREM A (F').** *Proof of A4 from A8, A8'. (Real domain.)*

By A8,  $f(A - x_1, A - x_2, \dots, A - x_n) = A + f(-x_1, -x_2, \dots, -x_n)$ , whence, by A8',  $f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n)$ . This shows that Set A8 reduces to Set A4, and hence determines the arithmetic mean in the domain of reals. Unlike the other sets, however, this Set A8 cannot be used if the  $n$  quantities  $x_1, x_2, \dots, x_n$  are restricted to the domain of positive values.

## APPENDIX II. POSTULATES FOR THE ARITHMETIC MEAN IN THE COMPLEX DOMAIN

In the domain of complex quantities, Sets A1 and A8 are sufficient to determine the arithmetic mean. (See proofs below.) Further, if we form a new postulate

**POSTULATE A9.**  $f(A - x_1, A - x_2, \dots, A - x_n) = A - f(x_1, x_2, \dots, x_n)$ , then a Set A9, comprising Postulates I, II, A9, will also be sufficient, as proved below. (This postulate A9 is the same as A4 without the restriction  $x_i < 1$ .)

Hence, in the complex domain, we have three sets of independent postulates for the arithmetic mean, as follows:

Set	Postulates				Examples Used			
A1	A1	I	II	IV	0	A I	A II''	A IV
A8	A8	I	II	A8'	0	A I	A II''	0'
A9	A9	I	II		0	A I	A II''	

The new example here required is

Ex. A II''.  $f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) = x + iy$ , where  $x$  is the median of the  $x$ 's if  $n$  is odd (or the arithmetic mean of their median-pair if  $n$  is even), and  $y$  is the median of the  $y$ 's if  $n$  is odd (or the arithmetic mean of their median-pair if  $n$  is even).

This example satisfies Postulates A1, A8, A8', A9, I and IV, but fails on Postulate II.

The properties of all these examples for the complex domain are shown in the following table:

	A1	A8	A8'	A9	I	II	IV
Ex. 0	A1	—	A8'	—	I	II	IV
Ex. 0'	A1	A8	—	—	I	II	IV
Ex. A I	A1	A8	A8'	A9	—	II	IV
Ex. A II''	A1	A8	A8'	A9	I	—	IV
Ex. A IV	A1	—	A8'	—	I	II	—

On the other hand, it is interesting to note that if the  $n$  quantities  $x_1, x_2, \dots, x_n$  are allowed to take on complex values, then Sets A2, A3, A4, A5, A6, A7, A6'', A7'' are not sufficient to determine the arithmetic mean, as is shown by the following examples:

Ex. A IX.

$f(x_1 + iy, x_2 + iy, \dots, x_n + iy) = (x_1 + iy + x_2 + iy + \dots + x_n + iy)/n$  when all the  $y$ 's are equal; otherwise,

$$f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) = (x_1 + x_2 + \dots + x_n)/n.$$

This example satisfies Postulates I, II, III, IV, III', IV', IV'', V, A2, A3, A4, but it is not the arithmetic mean. (This example would still satisfy A2 and A3, even if we removed the restrictions  $a < 1$ ,  $b < 1$ , and  $x_i < 1$ .)

Ex. A X. When  $n=2$ ,  $f(x_1, x_2) = (x_1 + x_2)/2$ ; when  $n > 2$ ,

$$f() = (x_1 + x_2 + \dots + x_n)/n \text{ or } |(x_1 + x_2 + \dots + x_n)/n|,$$

according as  $(x_1 + x_2 + \dots + x_n)/n$  is real or imaginary.

This example satisfies Postulates A1, I, II, III, IV', IV'', V, but fails on Postulate IV.

The proofs of the theorems in the complex domain, on account of their great simplicity, are here given in full, as follows:

THEOREM A (A''). *Proof of A from A1, I, II, IV. (Complex domain.)*

By II and A1,  $f(x_1, x_2, \dots, x_n) = f(0, x_1 + x_2, x_3, \dots, x_n)$ , since each side equals  $f\{(x_1 + x_2)/2, (x_1 + x_2)/2, x_3, \dots, x_n\}$ .

By successive applications of this result,

$$f(x_1, x_2, \dots, x_n) = f(0, 0, \dots, [x_1 + x_2 + \dots + x_n]).$$

Now let  $a = (x_1 + x_2 + \dots + x_n)/n$ , and take each  $x$  equal to  $a$ . Then  $f(a, a, \dots, a) = f(0, 0, \dots, 0, na) = a$ , by IV.

But also  $f(x_1, x_2, \dots, x_n) = f(0, 0, \dots, 0, na)$ .

Hence  $f(x_1, x_2, \dots, x_n) = (x_1 + x_2 + \dots + x_n)/n$ .

This proof shows that Set A1 determines the arithmetic mean, for all complex values of the variables.

**THEOREM A (F'').** *Proof of A1 and IV from A8, A8', I, II. (Complex domain.)*

Putting each  $x=0$  in A8', we have  $f(0, 0, \dots, 0)=0$ ; hence putting each  $x=0$  and  $A=a$ , in A8, we have  $f(a, a, \dots, a)=a$ , which is Postulate IV.

Again, putting  $A=a+b$ ,  $x_1=-a$ ,  $x_2=-b$ , and  $n=2$ , in A8, we have  $f(b, a) = a+b+f(-a, -b)$ , whence, by I and A8',  $f(a, b) = a+b-f(a, b)$ , whence  $f(a, b) = (a+b)/2$ , which is Postulate A1.

This proof shows that Set A8 reduces to Set A1, for all complex values.

**THEOREM A (G).** *Proof of A1 and IV from A9, I, II. (Complex domain.)*

In A9, put each  $x$  equal to 0, and  $A=0$ ; then

$$f(0, 0, \dots, 0) = -f(0, 0, \dots, 0),$$

whence  $f(0, 0, \dots, 0)=0$ .

Hence, putting each  $x$  equal to  $a$ , and  $A=a$ , we have

$$f(0, 0, \dots, 0) = a - f(a, a, \dots, a),$$

whence  $f(a, a, \dots, a)=a$ , which is Postulate IV for all values of  $a$ .

Again, put  $A=a+b$ ,  $x_1=a$ ,  $x_2=b$ , and  $n=2$ , in A9; then

$$f(b, a=a) + b - f(a, b),$$

whence, by I,  $f(a, b) = a+b-f(a, b)$ , whence  $f(a, b) = (a+b)/2$ , which is Postulate A1 for all values of  $a$ .

This proof shows that Set A9 reduces to Set A1, for all complex values.

The sufficiency of each of the Sets A1, A8, A9, is thus established.

*In conclusion, it may be noted that of all the known sets of postulates for the arithmetic mean, the only ones that are equally available in the positive domain, the real domain, and the complex domain, are Sets A1 and A9. Of these, Set A9 (consisting of Postulates A9, I, II) would appear to be the simplest.*

HARVARD UNIVERSITY,  
CAMBRIDGE, MASS.

## IRREGULAR DIFFERENTIAL SYSTEMS OF ORDER TWO AND THE RELATED EXPANSION PROBLEMS\*

BY  
M. H. STONE

We have previously discussed the similarities between the series of Fourier and of Birkhoff.† Since a series of Birkhoff is defined by a linear homogeneous differential system of the  $n$ th order in which the boundary conditions are of *regular* type,‡ it is natural to attempt an extension of the methods there employed to some systems with *irregular* boundary conditions. We shall discuss here the case  $n = 2$ , with the hope of giving a comparatively exhaustive treatment of the narrowed topic. From our point of view, it is not essential in this discussion that a series be treated with regard to its convergence: a sum by appropriate means we consider equally valuable. A treatment of the convergence of the formal expansions for a function restricted to have a certain number of derivatives and to satisfy certain boundary conditions has come to our attention since the completion of this paper.§ As Professor Jackson has suggested to the writer, the methods of Wilder in a similar problem could be applied to this end, as is obvious from a comparison of the formulas of this paper with his.|| It should be noted, however, that under our discussion of systems of type 1, Case I, the series for the function 1 can be seen to be divergent, so that such results are not so useful as it might appear. We note that the series discussed in this paper are entirely different from those discussed by Jackson and Hopkins in the case  $n \geq 3$ .¶

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† Stone, these Transactions, vol. 28 (1926), pp. 695-761.

‡ Birkhoff, these Transactions, vol. 9 (1908), p. 383.

§ Pollaczek-Geiringer, Mathematische Annalen, vol. 90 (1923), pp. 292-317.

|| C. E. Wilder, these Transactions, vol. 18 (1917), pp. 415-442.

¶ Hopkins, these Transactions, vol. 20 (1919), pp. 245-259; Jackson, Proceedings of the American Academy of Arts and Sciences, vol. 51 (1915-1916), pp. 383-417.

## I. CLASSIFICATION OF THE BOUNDARY CONDITIONS

Our first task is clearly that of separating all possible boundary conditions,  $n=2$ , into regular and irregular types. We shall refer to Birkhoff's memoir, cited above, for the definition of regularity, and for the fact that in the case where  $n$  is even the several conditions of that definition reduce to one. We then prove

THEOREM I. *The only irregular boundary conditions for  $n=2$  are of the form*

$$(1) \quad \begin{aligned} u'(0) + Au'(1) + Bu(1) &= 0, \\ u(0) - Au(1) &= 0, \end{aligned}$$

or of the form

$$(2) \quad \begin{aligned} Au'(0) - u'(1) + Bu(0) &= 0, \\ Au(0) + u(1) &= 0, \end{aligned}$$

where  $A, B$  are real or complex constants.\* The two differential systems obtained by adjoining to the equation

$$u'' + (\lambda + g)u = 0, \quad g(x) \text{ summable}, \quad 0 \leq x \leq 1,$$

the boundary conditions (1) and (2) respectively, are adjoint systems.

The general boundary conditions,  $n=2$ , can be written

$$\begin{aligned} a_1 u'(0) + b_1 u'(1) + c_1 u(0) + d_1 u(1) &= 0, \\ a_2 u'(0) + b_2 u'(1) + c_2 u(0) + d_2 u(1) &= 0, \end{aligned}$$

where  $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$  are any real or complex constants such that the linear forms of which they are the coefficients remain linearly independent. We must consider several cases.

Case I.  $a_1 b_2 - a_2 b_1 \neq 0$ . Employing the notation of Birkhoff's definition of regularity, we write

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s} \equiv \begin{vmatrix} (a_1 + b_1 s)i & -\left(a_1 + \frac{b_1}{s}\right)i \\ (a_2 + b_2 s)i & -\left(a_2 + \frac{b_2}{s}\right)i \end{vmatrix} \equiv \left(\frac{1}{s} - s\right)(a_1 b_2 - a_2 b_1).$$

Since  $\theta_1 \theta_2 \neq 0$ , the boundary conditions are regular.

\* Tamarkin, *Rendiconti del Circolo Matematico di Palermo*, vol. 34 (1912), pp. 360-361.

Case II.  $a_1b_2 - a_2b_1 = 0$ ;  $a_1$  and  $b_1$  not both zero. The boundary conditions can be reduced by linear combination to the form

$$\begin{aligned} a_1u'(0) + b_1u'(1) + c_1u(0) + d_1u(1) &= 0, \\ c_2u(0) + d_2u(1) &= 0. \end{aligned}$$

Then

$$\begin{aligned} \theta_0 + \theta_1s + \theta_2/s &\equiv \begin{vmatrix} i(a_1 + b_1s) & -i(a_1 + b_1/s) \\ (c_2 + d_2s) & (c_2 + d_2/s) \end{vmatrix} \equiv 2(a_1c_2 + b_1d_2) \\ &\quad + (a_1d_2 + b_1c_2)(1/s + s). \end{aligned}$$

If  $a_1d_2 + b_1c_2 \neq 0$ , the conditions are regular; on the other hand, if  $a_1d_2 + b_1c_2 = 0$ , the conditions are irregular. In this latter case, since  $c_2$  and  $d_2$  cannot both vanish, we have left

$$\begin{aligned} a_1u'(0) + b_1u'(1) + c_1u(0) + d_1u(1) &= 0, \\ a_1u(0) - b_1u(1) &= 0. \end{aligned}$$

If these boundary conditions have  $a_1 \neq 0$  they can be reduced to the form (1); if  $b_1 \neq 0$ , to the form (2).

Case III.  $a_1 = a_2 = b_1 = b_2 = 0$ . Since the boundary conditions

$$\begin{aligned} c_1u(0) + d_1u(1) &= 0, \\ c_2u(0) + a_2u(1) &= 0 \end{aligned}$$

are linearly independent they reduce to  $u(0) = u(1) = 0$ , a well known regular case.

It is now a matter of simple computation to show that the two differential systems defined in the latter part of the theorem are actually adjoint. The method employed is sufficiently familiar that we omit details.\*

## II. THE IRREGULAR BOUNDARY VALUE PROBLEM

We are now prepared to consider the boundary value problem in the irregular cases of Theorem I; that is, to investigate the values of  $\lambda$  for which the differential systems have solutions not identically zero. As is well known, these characteristic values are the same for a system and its adjoint. On setting  $\lambda = \rho^2$ , the two systems are

$$\begin{aligned} u'' + (\rho^2 + g)u &= 0, & u'' + (\rho^2 + g)u &= 0, \\ u'(0) + Au'(1) + Bu(1) &= 0, & Au'(0) - u'(1) + Bu(0) &= 0, \\ u(0) - Au(1) &= 0, & Au(0) + u(1) &= 0. \end{aligned}$$

\* Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, Chapter II.

The characteristic values in  $\rho$  may be found as the roots of the equation

$$\begin{vmatrix} u_1'(0) + Au_1'(1) + Bu_1(1) & u_2'(0) + Au_2'(1) + Bu_2(1) \\ u_1(0) - Au_1(1) & u_2(0) - Au_2(1) \end{vmatrix} = 0$$

where  $u_1, u_2$  are any two linearly independent solutions of the differential equation defined for all values of  $\rho$ . The existence of such solutions has been demonstrated in our preceding paper.

A number of outstanding facts are revealed at once if we choose  $u_1, u_2$  as solutions satisfying the boundary conditions

$$u_1(0) = 1, u_1'(0) = 0, u_2(0) = 0, u_2'(0) = 1.$$

On expanding the determinant and making use of the fact that

$$u_1(1)u_2'(1) - u_1'(1)u_2(1) = u_1(0)u_2'(0) - u_1'(0)u_2(0) = 1,$$

we find the equation

$$(A^2 - 1) + A(u_1(1) - u_2'(1)) - Bu_2(1) = 0.$$

If  $A = B = 0$ , there can be no characteristic values; henceforth this possibility shall be excluded. Again, if  $A^2 - 1 = 0, B = 0$ , the equation is  $u_1(1) - u_2'(1) = 0$ , whence we conclude that a root in this case cannot be a root in the case  $A^2 - 1 \neq 0, B = 0$ . If we consider the two differential systems when  $A^2 - 1 = 0, B = 0$ , one of these is

$$u'' + (\rho^2 + g)u = 0,$$

$$u'(0) + u'(1) = 0,$$

$$u(0) - u(1) = 0.$$

If we suppose that the equation  $g(x) = g(1-x)$  is satisfied almost everywhere on  $(0, 1)$ , the function satisfying the differential system

$$u'' + (\rho^2 + g)u = 0,$$

$$u(\frac{1}{2}) = 1, u'(\frac{1}{2}) = 0,$$

is a solution of the irregular system above for all values of  $\rho$ . Thus it is evident that under appropriate circumstances all values of  $\rho$  are characteristic values, while under others there is no characteristic value; in such cases there is no expansion problem.

Having thus obtained a view of some of the peculiarities which can arise in the irregular boundary value problem, we can continue our discussion, under hypotheses which enable us to make more definite assertions. We shall require in all our succeeding work that  $g(x)$  be continuous together

with its derivatives of all orders on the interval  $(0, 1)$ , although we could in many instances lighten this restriction. Then, as we saw in Theorem III' of our previous paper, there exist on the first quadrant of the  $\rho$ -plane solutions  $u_1, u_2$  of  $u'' + (\rho^2 + g)u = 0$  with the asymptotic forms

$$\begin{aligned} u_1 &= e^{\rho i x} \left( 1 + \sum_{l=1}^{l=m} \frac{A_{10}(x)}{(\rho i)^l} + \frac{E_{10}(x, \rho)}{\rho^{m+1}} \right), \\ u_1' &= \rho i e^{\rho i x} \left( 1 + \sum_{l=1}^{l=m} \frac{A_{11}(x)}{(\rho i)^l} + \frac{E_{11}(x, \rho)}{\rho^{m+1}} \right), \\ u_2 &= e^{-\rho i x} \left( 1 + \sum_{l=1}^{l=m} \frac{A_{10}(x)}{(-\rho i)^l} + \frac{E_{20}(x, \rho)}{\rho^{m+1}} \right), \\ u_2' &= -\rho i e^{-\rho i x} \left( 1 + \sum_{l=1}^{l=m} \frac{A_{11}(x)}{(-\rho i)^l} + \frac{E_{21}(x, \rho)}{\rho^{m+1}} \right), \end{aligned}$$

the functions  $E$  being uniformly bounded,  $0 \leq x \leq 1$ , for  $\rho$  on the first quadrant. In particular,  $A_{10}(x) = A_{11}(x)$ . Similarly, on the fourth quadrant there exist solutions  $u_1, u_2$  whose asymptotic forms are given by replacing  $\rho$  by  $-\rho$  in the exponential terms and sums appearing in the forms given above for the first quadrant; the functions  $E$  are, of course, not necessarily the same.

**THEOREM II.** *If  $u_1, u_2$  are the functions defined above for the first quadrant, then for any positive integral  $m$*

$$\begin{aligned} D &\equiv \begin{vmatrix} u_1'(0) + Au_1'(1) + Bu_1(1) & u_2'(0) + Au_2'(1) + Bu_2(1) \\ u_1(0) - Au_1(1) & u_2(0) - Au_2(1) \end{vmatrix} \\ &= 2(1 - A^2)\rho i \left( 1 + \frac{E_1(\rho)}{\rho^2} \right) \\ &\quad + A \left( e^{\rho i} \left( \sum_{l=1}^{l=m-1} \frac{\alpha_l}{\rho^l} + \frac{E_2(\rho)}{\rho^m} \right) + e^{-\rho i} \left( \sum_{l=1}^{l=m-1} \frac{(-1)^{l+1}\alpha_l}{\rho^l} + \frac{E_3(\rho)}{\rho^m} \right) \right) \\ &\quad + B \left( e^{\rho i} \left( 1 + \sum_{l=1}^{l=m} \frac{\beta_l}{\rho^l} + \frac{E_4(\rho)}{\rho^{m+1}} \right) + e^{-\rho i} \left( 1 + \sum_{l=1}^{l=m} \frac{(-1)^{l+1}\beta_l}{\rho^l} + \frac{E_5(\rho)}{\rho^{m+1}} \right) \right), \end{aligned}$$

$\alpha_1, \dots, \alpha_{m-1}, \beta_1, \dots, \beta_m$  being constants, and the functions  $E$  being bounded and analytic on the first quadrant. On the fourth quadrant the substitution of the corresponding functions  $u_1, u_2$  gives an analogous formula involving the same numbers  $\alpha, \beta$ .

We shall carry through the computations for the first quadrant; by replacing  $\rho$  by  $-\rho$  in the exponential terms and sums appearing in the expression thus obtained we find the asymptotic form for the fourth quadrant. On expanding the determinant and recalling that

$$u_1'(0)u_2(0) - u_1(0)u_2'(0) = u_1'(1)u_2(1) - u_1(1)u_2'(1),$$

we find as the result

$$\begin{aligned} (1 - A^2)(u_1'(0)u_2(0) - u_1(0)u_2'(0)) + A\{ & ((u_1'(1)u_2(0) + u_1(1)u_2'(0)) \\ & - (u_2'(1)u_1(0) + u_2(1)u_1'(0))) \} \\ & + B(u_1(1)u_2(0) - u_1(0)u_2(1)). \end{aligned}$$

By direct substitution of the asymptotic forms for  $u_1, u_2$ , in which  $A_{10}(x) = A_{11}(x)$ , as we noted above, we find

$$\begin{aligned} u_1'(0)u_2(0) - u_1(0)u_2'(0) &= 2\rho i \left( 1 + \frac{E_1(\rho)}{\rho^2} \right), \\ u_1'(1)u_2(0) + u_1(1)u_2'(0) &= e^{\rho i} \left( \sum_{l=1}^{l=m-1} \frac{\alpha_l}{\rho^l} + \frac{E_2(\rho)}{\rho^m} \right), \\ u_1(1)u_2(0) &= e^{\rho i} \left( 1 + \sum_{l=1}^{l=m} \frac{\beta_l}{\rho^l} + \frac{E_4(\rho)}{\rho^{m+1}} \right). \end{aligned}$$

If we neglect the  $E$  terms in  $u_1, u_2$ , the replacement of  $\rho$  by  $-\rho$  interchanges  $u_1, u_2$ . Hence by replacing  $\rho$  by  $-\rho$  in the second and third of the asymptotic forms just obtained, we show that

$$\begin{aligned} u_2'(1)u_1(0) + u_2(1)u_1'(0) &= e^{-\rho i} \left( \sum_{l=1}^{l=m-1} \frac{(-1)^l \alpha_l}{\rho^l} + \frac{E_3(\rho)}{\rho^m} \right), \\ u_2(1)u_1(0) &= e^{-\rho i} \left( 1 + \sum_{l=1}^{l=m} \frac{(-1)^l \beta_l}{\rho^l} + \frac{E_5(\rho)}{\rho^{m+1}} \right). \end{aligned}$$

When the five asymptotic forms are substituted in the expanded determinant, the theorem is obtained.

As a consequence of Theorem II, we lay down

DEFINITION I. *The irregular differential system*

$$\begin{aligned} u'' + (\lambda + g)u &= 0, \quad \lambda = \rho^2, \\ u'(0) + Au'(1) + Bu(1) &= 0, \\ u(0) - Au(1) &= 0, \end{aligned}$$

where  $g(x)$  is continuous together with its derivatives of all orders,  $0 \leq x \leq 1$ , shall be termed of type 1 if  $B \neq 0$ ; and a system of type  $M$ ,  $M \geq 2$ , if  $B = 0$  and  $\alpha_{M-1}$  is the first of the  $\alpha$ 's different from zero. If  $B = 0$ ,  $\alpha_1 = 0$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 0, \dots$ , the system shall be termed of type  $\Omega$ .

We have found no means of investigating systems of type  $\Omega$ ; that they exist is shown by the examples given above.

In the case of systems of type  $M$ ,  $M = 1, 2, \dots$ , we can give a complete discussion of the distribution of characteristic values.

**THEOREM III.** *The irregular differential systems of the second order of finite type  $M$  have infinitely many characteristic values in  $\rho$  when  $A^2 - 1 = 0$ . They are distributed asymptotically near the roots of  $e^{\rho i} + (-1)^M e^{-\rho i} = 0$ ; for large  $|\rho|$  there is one simple characteristic value near each root of this equation.*

We recall that the asymptotic forms of  $u_1, u_2$  on a given quadrant are valid also in a sector including the quadrant and bounded by rays parallel to the two axes. Hence this is also true of the asymptotic forms derived from those for  $u_1, u_2$ . Thus the characteristic values of  $\rho$  on the first quadrant and within a specified distance of it are found as roots of the equation

$$B(e^{\rho i}[1] - e^{-\rho i}[1]) = 0, \quad M = 1,$$

$$A\alpha_{M-1}(e^{\rho i}[1] + (-1)^M e^{-\rho i}[1])/\rho^{M-1} = 0, \quad M \geq 2,$$

or of

$$e^{\rho i}[1] + (-1)^M e^{-\rho i}[1] = 0 \quad (M = 1, 2, \dots).$$

Similarly the characteristic values on the fourth quadrant and within a specified distance of it are found as the roots of

$$e^{\rho i}[1] + (-1)^M e^{-\rho i}[1] = 0 \quad (M = 1, 2, \dots).$$

The characteristic values on the left half-plane are the negatives of those on the right. Now by methods employed in the discussion of the regular boundary value problem,  $n=2$ , the roots of the equations  $e^{\rho i}[1] + [-1]^M e^{-\rho i} = 0$  on the two quadrants respectively have the asymptotic distribution described.\*

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\* Birkhoff, these Transactions, vol. 9 (1908), pp. 386-387; Tamarkin, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 353-358; Birkhoff, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 116-118.

**THEOREM IV.** *The irregular differential systems of the second order of finite type  $M$  have infinitely many characteristic values in  $\rho$  when  $A^2 - 1 \neq 0$ . If we write  $\mu = 2(A^2 - 1)i/B$ ,  $M = 1$ , and  $\mu = 2(A^2 - 1)i/\alpha_{M-1}$ ,  $M \geq 2$ , the asymptotic distribution of these roots on the right half-plane is as follows:*

(1) *for  $|\rho|$  sufficiently large on the first quadrant, there is one simple characteristic value near each of the points  $\rho = r + is$ ,*

$$r = \arccos \frac{(-1)^{M+1} \mu_1}{\sqrt{\mu_1^2 + \mu_2^2}} = \arcsin \frac{(-1)^M \mu_2}{\sqrt{\mu_1^2 + \mu_2^2}},$$

$$s = \log(\sqrt{\mu_1^2 + \mu_2^2} r^M), \quad \mu = \mu_1 + i\mu_2;$$

(2) *for  $|\rho|$  sufficiently large on the fourth quadrant, there is one simple characteristic value near each of the points*

$$r = \arccos \frac{-\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}} = \arcsin \frac{-\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}},$$

$$s = -\log(\sqrt{\mu_1^2 + \mu_2^2} r^M), \quad \mu = \mu_1 + i\mu_2.$$

*The characteristic values on the left half-plane are the negatives of those on the right.*

We shall limit our discussion to the characteristic values on the first quadrant; the treatment of those on the fourth quadrant is entirely analogous.

By Theorem II the characteristic values on the first quadrant are found as the roots of

$$\mu\rho[1] + \frac{e^{\rho i}[1] + (-1)^M e^{-\rho i}[1]}{\rho^{M-1}} = 0,$$

or, after multiplication by  $(-1)^M \rho^{M-1}$ , of

$$e^{-\rho i}[1] + \mu(-1)^M \rho^M \left( [1] + \frac{e^{\rho i}[1]}{\mu \rho^M} \right) = 0.$$

This last equation takes the form

$$e^{-\rho i} = (-1)^{M+1} \mu \rho^M [1]$$

on the first quadrant.

If there are infinitely many roots  $\rho = r + is$  on the first quadrant, then  $\lim_{|\rho| \rightarrow \infty} (s/r) = 0$ . To prove this statement it is sufficient to show that there

exists no positive  $\epsilon$  for which infinitely many roots satisfy  $s/r \geq \epsilon$ . If such an  $\epsilon$  exists,  $s \rightarrow \infty$  as  $|\rho| \rightarrow \infty$ ; and for the roots in question

$$e^s(\cos r - i \sin r) = (-1)^{M+1} \mu (r + is)^M [1],$$

whence

$$1 \leq |\mu| |[1]| |re^{-s/M} + is e^{-s/M}|^M \leq |\mu| |[1]| |(s/\epsilon) e^{-s/M} + is e^{-s/M}|^M \rightarrow 0$$

as  $|\rho| \rightarrow \infty$ . The contradiction shows that  $\epsilon$  does not exist. Hence

$$\lim_{|\rho| \rightarrow \infty} (s/r) = 0.$$

Now we find

$$\frac{e^s}{|\mu| r^M} = |1 + i(s/r)|^M |[1]|,$$

whence

$$s - \log(\sqrt{\mu_1^2 + \mu_2^2} r^M) = M \log |1 + i(s/r)| + \log |[1]| = \epsilon_{|\rho|} \rightarrow 0$$

as  $|\rho| \rightarrow \infty$ . Using this relation we obtain

$$\sqrt{\mu_1^2 + \mu_2^2} (\cos r - i \sin r) = (-1)^{M+1} \mu (1 + i(s/r))^M [1] e^{-i\rho} \rightarrow (-1)^{M+1} \mu$$

as  $|\rho| \rightarrow \infty$ ; thus

$$r = \arccos \frac{(-1)^{M+1} \mu_1}{\sqrt{\mu_1^2 + \mu_2^2}} + \epsilon'_{|\rho|},$$

$$r = \arcsin \frac{(-1)^M \mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} + \epsilon'_{|\rho|},$$

where  $\epsilon'_{|\rho|} \rightarrow 0$  as  $|\rho| \rightarrow \infty$ . In short, if there exist infinitely many characteristic values on the first quadrant, they necessarily lie asymptotically near the points described in (1).

We shall now let  $\rho'$  be one of the points described under (1). In the function  $e^{-\rho i} + (-1)^M \mu \rho^M [1]$ , analytic in  $\rho$  on the first quadrant, we replace  $\rho$  by  $\rho' + \xi$ ; we restrict  $\xi$  to the circle  $|\xi| \leq \epsilon' < 2\pi$ . There results a function analytic in  $\xi$ ,

$$e^{-\rho' i} e^{-i\xi} + (-1)^M \mu (\rho' + \xi)^M [1] \equiv (-1)^{M+1} \mu \rho' e^{-i\xi} + (-1)^M \mu (\rho' + \xi)^M [1].$$

If we divide by  $(-1)^M \mu \rho'^M \neq 0$  we obtain a new function,

$$1 - e^{-i\xi} + \epsilon(\xi, r', s'),$$

where  $\epsilon(\xi, r', s')$  is analytic in  $\xi$ ,  $|\xi| \leq \epsilon'$ , for all  $r', s'$ , and  $\lim_{|\rho'| \rightarrow \infty} \epsilon = 0$  uniformly,  $|\xi| \leq \epsilon'$ . If  $\xi$  describes the circle  $|\xi| = \epsilon'' \leq \epsilon'$  the argument of  $1 - e^{-i\xi}$  changes by  $2\pi$ , while  $|1 - e^{-i\xi}| \geq \eta' > 0$ . If we choose  $|\rho'|$  so large

that  $|\epsilon(\xi, r', s')| \leq \eta'/2$ , the argument of the function  $1 - e^{-i\xi} + \epsilon(\xi, r', s')$  also changes by  $2\pi$  when  $\xi$  describes the circle  $|\xi| = \epsilon''$ . In other words,  $1 - e^{-i\xi} + \epsilon$  has a simple zero in the circle  $|\xi| = \epsilon''$  for all  $\rho'$  such that  $|\rho'| \geq R'$ . Thus for large  $|\rho|$ , there exists one and only one simple characteristic value near each of the points described in (1).

### III. THE IRREGULAR EXPANSION PROBLEMS

We shall now make a comparative study of the differential systems

$$\begin{aligned} u'' + (\lambda + g)u &= 0, & u'' + \lambda u &= 0, \\ u'(0) + Au'(1) + Bu(1) &= 0, & u'(0) - u'(1) &= 0, \\ u(0) - Au(1) &= 0, & u(0) - u(1) &= 0. \end{aligned}$$

The other irregular differential system, adjoint to the first system given here, can be reduced to the form of the first by the substitution  $\bar{x} = 1 - x$  and therefore does not require separate consideration. We assume that the irregular differential system is of finite type. Then there exists for it a Green's function  $G(x, y; \lambda)$ ; the second differential system has a Green's function  $\bar{G}(x, y; \lambda)$  and gives rise to the expansion problem of Fourier.

We let  $\Sigma'$  denote the right half-plane for the complex variable  $\rho$ ,  $\lambda = \rho^2$ , from which the characteristic values of the irregular differential system and of the Fourier system have been removed, the interior of a small circle  $\sigma$  of radius  $\epsilon$  described about each such value as center being deleted. We denote by  $S'_I, S'_{IV}$  the parts of  $\Sigma'$  on the first and fourth quadrants respectively. We denote by  $\Gamma$  any circular arc with center at  $\rho = 0$  and central angle  $\pi$ , lying on  $\Sigma'$ . The image of  $\Gamma$  in the  $\lambda$ -plane will be a circle  $C$ ; the totality of such circles  $C$  forms an infinite set of concentric annular regions none of which contains a characteristic value of either differential system. From the behavior of the characteristic values for large  $|\lambda|$ , the circles  $C_0, C_1, C_2, \dots$ , as described in § II of our previous paper, can be selected from among the circles  $C$ ; this is true simultaneously for the two differential systems we are discussing. Finally, we let  $\gamma_I, \gamma_{IV}$  be the portions of  $\Gamma$  on the first and fourth quadrants respectively, while their common radius is  $R$ .

We shall study the behavior of

$$\frac{1}{2\pi i} \frac{\partial^k}{\partial x^k} \int_0^1 f(y) \int_{C_\nu} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} G(x, y; \lambda) d\lambda dy,$$

where  $f(x)$  is summable on  $(0, 1)$ ,  $l \geq M$ ,  $k = 0, 1, 2, \dots$ , as  $\nu \rightarrow \infty$ . The meaning of this integral is discussed at length in our preceding paper, §§ III and IV.

For this discussion we use Lemmas III, V', VI' of that paper, as well as Theorem IV. In addition we need two other lemmas.

LEMMA I. If  $A^2 - 1 = 0$  in an irregular differential system of order two and type  $M$ ,

$$\frac{1}{D} = \pm \frac{\rho^{M-1}}{C_M(e^{\rho i}[1] + (-1)^M e^{-\rho i}[1])}$$

on  $S'_I$  and  $S'_{IV}$  respectively, where  $C_1 = B \neq 0$  and  $C_M = A\alpha_{M-1} \neq 0$ ,  $M \geq 2$ . Furthermore,

$$\frac{1}{|e^{2\rho i}[1] + (-1)^M[1]|} \leq K, \quad \rho \text{ on } S'_I,$$

$$\frac{1}{|[1] + (-1)^M e^{-2\rho i}[1]|} \leq K, \quad \rho \text{ on } S'_{IV}.$$

The first part of the lemma is virtually a restatement of Theorem II. The second part is proved in exactly the same way as the corresponding facts in the case of regular differential systems.\*

LEMMA II. If  $A^2 - 1 \neq 0$  in an irregular differential system of order two and type  $M$ ,

$$\frac{1}{D} = \frac{\pm \rho^{M-1}}{C_M(\mu \rho^M[1] + e^{\rho i}[1] + (-1)^M e^{-\rho i}[1])},$$

where  $\mu = 2(A^2 - 1)i/C_M$ , on  $S'_I$  and  $S'_{IV}$  respectively. Furthermore,

$$\frac{1}{|e^{2\rho i}[1] + \mu \rho^M e^{\rho i}[1] + (-1)^M[1]|} \leq K, \quad \rho \text{ on } S'_I,$$

$$\frac{1}{|e^{-2\rho i}(-1)^M[1] + \mu \rho^M e^{-\rho i}[1] + [1]|} \leq K, \quad \rho \text{ on } S'_{IV}.$$

The first part of the lemma is a restatement of Theorem II.

In order to prove that  $|e^{2\rho i}[1] + \mu \rho^M e^{\rho i}[1] + (-1)^M[1]| \geq 1/K > 0$  on  $S'_I$  we show that the equality  $e^{2\rho i}[1] + \mu \rho^M e^{\rho i}[1] + (-1)^M[1] = \eta$ , where  $|\eta|$  is small, requires that  $\rho$  lie near one of the characteristic values on the first quadrant. We let  $\rho'$  be a value of  $\rho$  satisfying this equation, which we can write

$$e^{\rho' i} + \frac{(-1)^M[1] - \eta[1]}{\mu \rho'^M} = 0.$$

\* Birkhoff, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), p. 120.

In the function  $e^{2\rho i}[1] + \mu\rho^M e^{\rho i}[1] + (-1)^M[1] \equiv \mu\rho^M e^{\rho i}[1] + (-1)^M[1]$ , analytic in  $\rho$  on the first quadrant, we write  $\rho = \rho' + \xi$ ,  $|\xi| \leq \epsilon$ , where  $\epsilon$  is the radius of the circles  $\sigma$  described above. We obtain the function

$$- \frac{(-1)^M[1] - \eta[1]}{\rho'^M} (\rho' + \xi)^M e^{i\xi}[1] + (-1)^M[1],$$

analytic in  $\xi$ . On multiplying it by  $e^{-i\xi}(-1)^{M+1}/[1] \neq 0$ ,  $|\xi| \leq \epsilon$ , the term  $[1]$  being the last such term in the preceding expression, we find a function

$$1 - (-1)^{M+1}\eta - e^{-i\xi} + \zeta(\xi, \rho', \eta),$$

where  $\zeta$  is analytic in  $\xi$  for each pair of values  $\rho'$ ,  $\eta$ , and where also  $\lim_{|\rho'| \rightarrow \infty} \zeta(\xi, \rho', \eta) = 0$  uniformly,  $|\xi| \leq \epsilon$ ,  $|\eta| \leq H$ . For all  $|\eta| \leq \eta'$  where  $\eta'$  is sufficiently small

$$|1 - (-1)^{M+1}\eta - e^{-i\xi}| \geq \delta > 0, \quad |\xi| = \epsilon;$$

and  $\arg(1 - (-1)^{M+1}\eta - e^{-i\xi})$  changes by  $2\pi$  when  $\xi$  describes the circle  $|\xi| = \epsilon$ . We determine  $R'$  so that

$$|\zeta(\xi, \rho', \eta)| \leq \frac{\delta}{2}, \quad |\xi| = \epsilon, \quad |\eta| \leq \eta', \quad |\rho'| \geq R'.$$

Then  $\arg(1 - (-1)^{M+1}\eta - e^{-i\xi} + \zeta)$  changes by  $2\pi$  when  $\xi$  describes the circle  $|\xi| = \epsilon$ ; and the function itself vanishes in the circle. In other words, there is a characteristic value within distance  $\epsilon$  of  $\rho = \rho'$  if  $|\eta| \leq \eta'$ ,  $|\rho'| \geq R'$ ; and  $\rho'$  then lies in a circle  $\sigma$ . Hence, for  $|\rho| \geq R'$  on  $S'_I$  we have

$$|e^{2\rho i}[1] + \mu\rho^M e^{\rho i}[1] + (-1)^M[1]| > \eta',$$

as we were to show. The statement of the lemma follows at once. Similar reasoning applies on  $S'_{IV}$ .

We now demonstrate

**THEOREM V.** *For an irregular differential system of the second order of type M*

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_I} \left(1 - \frac{\rho^8}{R^8}\right)^{k+1} \left(2\rho \left\{\frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k}\right\} - \left\{F_{I,k}^0; F_{I,k}^1\right\}\right) d\rho dy = 0,$$

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_{IV}} \left(1 - \frac{\rho^8}{R^8}\right)^{k+1} \left(2\rho \left\{\frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k}\right\} - \left\{F_{IV,k}^0; F_{IV,k}^1\right\}\right) d\rho dy = 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ , where  $k=0, 1, 2, \dots, l \geq M$ ,  $f(x)$  is summable on  $(0, 1)$ , and

$$F_{I,k}^0(x, y, \rho) \equiv + \sum_{s=0}^{s=k} i^{s+1} \rho^s e^{\rho i(x-y)} \sum_{\alpha+\beta=k-s} A_{\alpha k}(x) B_{\beta}(y),$$

$$F_{I,k}^1(x, y, \rho) \equiv - \sum_{s=0}^{s=k} (-i)^{s+1} \rho^s e^{-\rho i(x-y)} \sum_{\alpha+\beta=k-s} A_{\alpha k}(x) B_{\beta}(y),$$

$$F_{IV,k}^0(x, y, \rho) \equiv - \sum_{s=0}^{s=k} (-i)^{s+1} \rho^s e^{-\rho i(x-y)} \sum_{\alpha+\beta=k-s} A_{\alpha k}(x) B_{\beta}(y),$$

$$F_{IV,k}^1(x, y, \rho) \equiv + \sum_{s=0}^{s=k} i^{s+1} \rho^s e^{\rho i(x-y)} \sum_{\alpha+\beta=k-s} A_{\alpha k}(x) B_{\beta}(y).$$

The functions  $A(x)$ ,  $B(y)$  are those defined in our preceding paper in Theorem III' and Lemma XIV. The expression

$$\frac{1}{2\pi i} \frac{\partial^k}{\partial x^k} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda_r^4}\right)^{k+l} G(x, y; \lambda) d\lambda dy,$$

$k=0, 1, 2, \dots, l \geq M$ , is therefore equivalent on any interval  $(a, b)$  to a linear combination with coefficients  $A_{\alpha k}(x)$  of means of order  $k+l$ , formed from the Fourier series and their derived series up to order  $k$  for the functions  $f(x)B_0(x) \equiv f(x)$ ,  $f(x)B_1(x)$ ,  $\dots$ ,  $f(x)B_k(x)$ . On any interval  $(a, b)$  the expansion problems associated with an irregular differential system of the second order of type  $M$  are thus phrased as problems in the theory of Fourier series. In particular

$$\lim_{r \rightarrow \infty} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda_r^4}\right)^l (G(x, y; \lambda) - \bar{G}(x, y; \lambda)) d\lambda dy = 0, \quad l \geq M,$$

uniformly,  $0 < a \leq x \leq b < 1$ .

On putting

$$W_1(u) \equiv u'(0) + Au'(1) + Bu(1),$$

$$W_2(u) \equiv u(0) - Au(1),$$

$$\tau_1 v_1(y) + \tau_2 v_2(y) \equiv \frac{\begin{vmatrix} u_1(y) & u_2(y) \\ \tau_1 & \tau_2 \end{vmatrix}}{\begin{vmatrix} u_1(y) & u_2(y) \\ u_1'(y) & u_2'(y) \end{vmatrix}},$$

we can write

$$2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} \equiv 2\rho \{ u_1^{(k)}(x)v_1(y); -u_2^{(k)}(x)v_2(y) \}$$

$$+ 2\rho \frac{\begin{vmatrix} u_1^{(k)}(x) & u_2^{(k)}(x) & 0 \\ W_1(u_1) & W_1(u_2) & +Au_1'(1)v_1(y) + Bu_1(1)v_1(y) - u_2'(0)v_2(y) \\ W_2(u_1) & W_2(u_2) & -Au_1(1)v_1(y) - u_2(0)v_2(y) \end{vmatrix}}{\begin{vmatrix} W_1(u_1) & W_1(u_2) \\ W_2(u_1) & W_2(u_2) \end{vmatrix}}.$$

Employing a familiar notation, we have on  $S_1'$

$$2\rho \{ u_1^{(k)}(x)v_1(y); -u_2^{(k)}(x)v_2(y) \} \equiv \{ F_{1,k}^0; F_{1,k}^1 \}$$

$$+ \{ e^{\rho i(x-y)} m_1(x, y, \rho)/\rho; e^{-\rho i(x-y)} m_2(x, y, \rho)/\rho \},$$

$$W_1(u_1) \equiv \rho i[1] + A\rho i e^{\rho i}[1] + B e^{\rho i}[1] \equiv \rho m(\rho),$$

$$W_1(u_2) \equiv -\rho i[1] - A\rho i e^{-\rho i}[1] + B e^{-\rho i}[1] \equiv \rho m(\rho) e^{-\rho i},$$

$$W_2(u_1) \equiv [1] - A e^{\rho i}[1] \equiv m(\rho),$$

$$W_2(u_2) \equiv [1] - A e^{-\rho i}[1] \equiv m(\rho) e^{-\rho i},$$

$$Au_1'(1)v_1(y) + Bu_1(1)v_1(y) - u_2'(0)v_2(y) \equiv \frac{A}{2} e^{\rho i(1-y)}[1] - \frac{Bi}{2\rho} e^{\rho i(1-y)}[1]$$

$$- \frac{1}{2} e^{\rho i y}[1] \equiv m(y, \rho),$$

$$-Au_1(1)v_1(y) - u_2(0)v_2(y) \equiv m(y, \rho)/\rho,$$

$$Au_1'(1) \int_{\alpha}^x v_1(y) dy + Bu_1(1) \int_{\alpha}^x v_1(y) dy - u_2'(0) \int_{\alpha}^x v_2(y) dy$$

$$\equiv m(x, \rho)/\rho, \quad 0 \leq \alpha \leq 1,$$

$$-Au_1(1) \int_{\alpha}^x v_1(y) dy - u_2(0) \int_{\alpha}^x v_2(y) dy \equiv \frac{m(x, \rho)}{\rho^2}, \quad 0 \leq \alpha \leq 1.$$

The last two results are consequent upon the fact that

$$\begin{aligned} e^{\rho i} \int_a^x v_1(y) dy &= \frac{-i}{2\rho} \int_a^x e^{\rho i(1-y)} [1] dy = \frac{-i}{2\rho} \frac{e^{\rho i(1-x)} - e^{\rho i(1-a)}}{-\rho i} \\ &\quad + \frac{-i}{2\rho} \int_a^x e^{\rho i(1-y)} [0] dy = m(x, \rho) / \rho^2, \\ \int_a^x v_2(y) dy &= \frac{i}{2\rho} \int_a^x e^{\rho i y} [1] dy = m(x, \rho) / \rho^2. \end{aligned}$$

Finally

$$\frac{1}{\begin{vmatrix} W_1(u_1) & W_1(u_2) \\ W_2(u_1) & W_2(u_2) \end{vmatrix}} = \frac{\rho^{M-1} e^{\rho i}}{C_M(e^{2\rho i}[1] + \mu \rho^M e^{\rho i}[1] + (-1)^M[1])} = \rho^{M-1} m(\rho) e^{\rho i}$$

by Lemmas I and II. Hence on  $S'_l$  we find

$$\begin{aligned} 2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} - \{F_{1,k}^0; F_{1,k}^1\} &\equiv \{e^{\rho i(x-y)} m_1(x, y, \rho) / \rho; e^{-\rho i(x-y)} m_2(x, y, \rho) / \rho\} \\ &\quad + \rho^{M+k} e^{\rho i x} m_3(x, y, \rho) + \rho^{M+k} e^{\rho i(1-x)} m_4(x, y, \rho), \\ \int_a^x \left( 2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} - \{F_{1,k}^0; F_{1,k}^1\} \right) dy \\ &\equiv \int_a^x \left\{ e^{\rho i(x-y)} \frac{m_1(x, y, \rho)}{\rho}; e^{-\rho i(x-y)} \frac{m_2(x, y, \rho)}{\rho} \right\} dy \\ &\quad + \rho^{M+k-1} e^{\rho i x} m_3(x, \rho) + \rho^{M+k-1} e^{\rho i(1-x)} m_4(x, \rho). \end{aligned}$$

By Lemmas III, V' of our antecedent paper,

$$\int_{\gamma_l} \left( 1 - \frac{\rho^8}{R^8} \right)^{k+l} \left( 2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} - \{F_{1,k}^0; F_{1,k}^1\} \right) d\rho, \quad l \geq M,$$

is uniformly bounded,  $0 < a \leq x \leq b < 1$ ,  $0 \leq y \leq 1$ , for all  $\gamma_l$  on  $S'_l$ . Lemmas V' and VI' show that

$$\lim_{R \rightarrow \infty} \int_a^x \int_{\gamma_l} \left( 1 - \frac{\rho^8}{R^8} \right)^{k+l} \left( 2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} - \{F_{1,k}^0; F_{1,k}^1\} \right) d\rho dy = 0, \quad l \geq M,$$

and also

$$\lim_{R \rightarrow \infty} \int_a^b \int_{\gamma_l} \left( 1 - \frac{\rho^8}{R^8} \right)^{k+l} \left( 2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^k G}{\partial x^k} \right\} - \{F_{1,k}^0; F_{1,k}^1\} \right) d\rho dy = 0, \quad l \geq M,$$

uniformly,  $0 < a \leq x \leq b < 1$ . An application of the theorem of Lebesgue\* quoted as Theorem IV in our previous paper shows that

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_I} \left(1 - \frac{\rho^8}{R^8}\right)^{k+l} \left(2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^l G}{\partial x^l} \right\} - \{F_{I,k}^0; F_{I,k}^1\}\right) d\rho dy = 0, \quad l \geq M,$$

uniformly,  $0 < a \leq x \leq b < 1$ .

Since  $u_1$  and  $u_2$  on  $S'_I$  are changed formally into  $u_1$  and  $u_2$  on  $S'_{IV}$  when  $\rho$  is replaced by  $-\rho$ , it is possible to obtain the asymptotic form for

$$2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^l G}{\partial x^l} \right\}$$

on  $S'_I$  by replacing  $\rho$  by  $-\rho$  in that established for  $S'_{IV}$ . Thus we see almost immediately that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_{IV}} \left(1 - \frac{\rho^8}{R^8}\right)^{k+l} \left(2\rho \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^l G}{\partial x^l} \right\} \right. \\ \left. - \{F_{IV,k}^0; F_{IV,k}^1\}\right) d\rho dy = 0, \quad l \geq M, \end{aligned}$$

uniformly,  $0 < a \leq x \leq b < 1$ .

By Lemma XIII of our preceding paper we have

$$\begin{aligned} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} \left\{ \frac{\partial^k G}{\partial x^k}; \frac{\partial^l G}{\partial x^l} \right\} d\lambda dy \\ = \frac{\partial^k}{\partial x^k} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} G d\lambda dy, \end{aligned}$$

and it is easy to complete the present theorem in a manner analogous to that used for Theorems XXXII and XXXII' of that paper. For  $k=0$ ,

$$F_{I,0}^0 = -ie^{\rho i(x-y)}, \quad F_{I,0}^1 = -ie^{-\rho i(x-y)},$$

$$F_{IV,0}^0 = +ie^{-\rho i(x-y)}, \quad F_{IV,0}^1 = +ie^{\rho i(x-y)}.$$

We know that

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_I} \left(1 - \frac{\rho^8}{R^8}\right)^l (\bar{G} - \{-ie^{\rho i(x-y)}; -ie^{-\rho i(x-y)}\}) d\rho dy = 0, \quad l \geq 0,$$

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_{IV}} \left(1 - \frac{\rho^8}{R^8}\right)^l (\bar{G} - \{ie^{-\rho i(x-y)}; ie^{\rho i(x-y)}\}) d\rho dy = 0, \quad l \geq 0,$$

\* Lebesgue, Annales de la Faculté des Sciences de Toulouse, (3), vol. 1 (1909), pp. 52-55.

uniformly,  $0 < a \leq x \leq b < 1$ , as the evaluation of these limits occurs in the proof sketched for Theorem IX' in our paper on the regular expansion problems. It is immediately evident that

$$\lim_{l \rightarrow \infty} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l (G - \bar{G}) d\lambda dy = 0, \quad l \geq M,$$

uniformly,  $0 < a \leq x \leq b < 1$ . This completes the proof of the theorem.

From various known properties of Fourier series and the term-by-term derivative series of Fourier series, we deduce the theorems which follow. The details of proof are strictly analogous to those given under the corresponding theorems of § VI of the paper on Birkhoff series.

THEOREM VI. *The expansions*

$$\frac{1}{2\pi i} \frac{\partial^k}{\partial x^k} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} G(x, y; \lambda) d\lambda dy, \quad l \geq M,$$

associated with an irregular differential system of the second order of type  $M$  are such that their behavior at  $x = x_0$  interior to  $(0, 1)$  is independent of the nature of the summable function  $f(x)$  outside an arbitrarily small neighborhood of  $x_0$ .

THEOREM VII. *If  $f(x)$  is summable,  $0 \leq x \leq 1$ ,*

$$\lim_{l \rightarrow \infty} \frac{1}{2\pi i} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} G(x, y; \lambda) d\lambda dy = f(x)$$

*almost everywhere,  $0 < x < 1$ , if  $l \geq M$ ; if  $f(x)$  is continuous the convergence is uniform,  $0 < a \leq x \leq b < 1$ .*

THEOREM VIII. *If  $\varphi(x)$  is a  $k$ -fold integral in the sense of Lebesgue,  $0 \leq x \leq 1$ , then*

$$\lim_{l \rightarrow \infty} \frac{1}{2\pi i} \frac{\partial^k}{\partial x^k} \int_0^1 \varphi(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^{k+l} G(x, y; \lambda) d\lambda dy = \varphi^{(k)}(x), \quad l \geq M,$$

*almost everywhere,  $0 < x < 1$ ; and if  $\varphi^{(k)}(x)$  is continuous the convergence is uniform,  $0 < a \leq x \leq b < 1$ .*

#### IV. THE IRREGULAR EXPANSION PROBLEMS OF TYPE 1

If in our differential system we take  $g(x) \equiv 0$ , there results a special system which is of type 1 or of type  $\Omega$ . Hence it is only in the case of systems of type 1,  $B \neq 0$ , that we can compare the systems

$$\begin{aligned}
u'' + (\lambda + g)u &= 0, & u'' + \lambda u &= 0, \\
u'(0) + Au'(1) + Bu(1) &= 0, & u'(0) + Au'(1) + Bu(1) &= 0, \\
u(0) - Au(1) &= 0, & u(0) - Au(1) &= 0.
\end{aligned}$$

If we denote by  $G, \bar{G}$  the Green's functions for these two differential systems respectively, it is our purpose to study the integral

$$\frac{1}{2\pi i} \int_0^1 \int_{C_r} f(y) (G(x, y; \lambda) - \bar{G}(x, y; \lambda)) d\lambda dy.$$

It will be seen to have the limit zero uniformly,  $0 < a \leq x \leq b < 1$ , as  $\nu \rightarrow \infty$ . We then make a special study of the second system, with interesting results.

The notations  $\rho, \Sigma', S'_I, S'_{IV}, \Gamma, \gamma_I, \gamma_{IV}, R, C_r$  have meanings entirely analogous to those in § III. We do not go into details.

We first prove

LEMMA III. On  $S'_I, (e^{2\rho i} + \mu\rho e^{\rho i} + (-1))/(e^{2\rho i}[1] + \mu\rho e^{\rho i}[1] + (-1)[1]) = [1]$ , and on  $S'_{IV}, (-e^{-2\rho i} + \mu\rho e^{-\rho i} + 1)/(-e^{-2\rho i}[1] + \mu\rho e^{-\rho i}[1] + [1]) = [1]$ .

If  $\mu = 0$  the lemma is Lemma VIII' of our preceding paper. If  $\mu \neq 0$  we recall that in the term  $\mu\rho e^{\rho i}[1], [1] = 1 + E_1(\rho)/\rho^2$ . Consequently

$$\begin{aligned}
\frac{e^{2\rho i} + \mu\rho e^{\rho i} + (-1)}{e^{2\rho i}[1] + \mu\rho e^{\rho i}[1] + (-1)[1]} &= 1 + \frac{e^{2\rho i}[0] + \mu\rho e^{\rho i}(1 - [1]) + [0]}{e^{2\rho i}[1] + \mu\rho e^{\rho i}[1] + (-1)[1]} \\
&= 1 + \frac{[0]}{e^{2\rho i}[1] + \mu\rho e^{\rho i}[1] + (-1)[1]} \\
&= [1],
\end{aligned}$$

by Lemma II, for all  $\rho$  on  $S'_I$ . Similar reasoning applies to the expression on  $S'_{IV}$ .

We can now obtain

THEOREM IX. If  $f(x)$  is summable on  $(0, 1)$ , then

$$\lim_{\nu \rightarrow \infty} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l (G(x, y; \lambda) - \bar{G}(x, y; \lambda)) d\lambda dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ .

We first show that

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_I} \left(1 - \frac{\rho^8}{R^8}\right)^l (G - \bar{G}) 2\rho d\rho dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ ; the result holds if  $\gamma_I$  is replaced by  $\gamma_{IV}$ , by reasoning whose details are now familiar; the theorem follows immediately.

We find by expanding the formula of Theorem V

$$2\rho G(x, y; \rho^2) \equiv 2\rho \{u_1(x)v_1(y); -u_2(x)v_2(y)\} \\ + \frac{2\rho e^{\rho i}}{B} \frac{\delta_{11}u_1(x)v_1(y) + \delta_{12}u_1(x)v_2(y) + \delta_{21}u_2(x)v_1(y) + \delta_{22}u_2(x)v_2(y)}{e^{2\rho i}[1] + \mu\rho e^{\rho i}[1] - [1]},$$

where

$$\delta_{11} = A^2(u_1'(1)u_2(1) - u_1(1)u_2'(1)) - A(u_1(1)u_2'(0) + u_1'(1)u_2(0)) - Bu_1(1)u_2(0) \\ = 2A^2i\rho[1] - Ae^{\rho i}[c] - Be^{\rho i}[1] = 2i\rho[A^2],$$

$$\delta_{12} = -A(u_2(1)u_2'(0) + u_2(0)u_2'(1)) - Bu_2(0)u_2(1) \\ = -2A i\rho e^{-\rho i}[1] - Be^{-\rho i}[1] = -2i\rho e^{-\rho i}[A],$$

$$\delta_{21} = A(u_1(1)u_1'(0) + u_1'(1)u_1(0)) + Bu_1(0)u_1(1) \\ = 2A i\rho e^{\rho i}[1] + Be^{\rho i}[1] = 2i\rho e^{\rho i}[A],$$

$$\delta_{22} = (u_2(0)u_1'(0) - u_2'(0)u_1(0)) + A(u_1'(1)u_2(0) + u_1(1)u_2'(0)) + Bu_1(1)u_2(0) \\ = 2i\rho[1] + Ae^{\rho i}[c] + Be^{\rho i}[1] = 2i\rho[1],$$

for  $\rho$  on  $S'_1$ . Using the asymptotic forms for  $u_1, u_2, v_1, v_2$  and the result of Lemma III we find

$$2\rho G(x, y; \rho^2) = \{ -ie^{\rho i(x-y)}[1]; -ie^{-\rho i(x-y)}[1] \} \\ + \frac{1}{B} \frac{e^{\rho i x} e^{\rho i(1-y)}[\Delta_{11}] + e^{\rho i x} e^{\rho i y}[\Delta_{12}] + e^{\rho i(1-x)} e^{\rho i(1-y)}[\Delta_{21}] + e^{\rho i(1-x)} e^{\rho i y}[\Delta_{22}]}{e^{2\rho i} + \mu\rho e^{\rho i} - 1},$$

where the terms  $[\Delta]$  in the numerator are of the forms

$$\Delta(x, \rho) \left( 1 + \frac{B_1(y)}{\rho i} + \frac{E_2(y, \rho)}{\rho^2} \right), \quad \Delta(x, \rho) \left( 1 - \frac{B_1(y)}{\rho i} + \frac{E_2(y, \rho)}{\rho^2} \right),$$

and where

$$\Delta_{11}(x, \rho) = 2\rho[A^2],$$

$$\Delta_{12}(x, \rho) = -2\rho[A],$$

$$\Delta_{21}(x, \rho) = 2\rho[A],$$

$$\Delta_{22}(x, \rho) = -2\rho[1].$$

In particular we have

$$2\rho\bar{G}(x, y; \rho^2) = \{ -ie^{\rho i(x-y)}; -ie^{-\rho i(x-y)} \} \\ + \frac{1}{B} \left( \frac{1}{e^{2\rho i} + \mu\rho e^{\rho i} - 1} \right) (\bar{\Delta}_{11}(x, \rho)e^{\rho i x}e^{\rho i(1-y)} + \bar{\Delta}_{12}(x, \rho)e^{\rho i x}e^{\rho i y} \\ + \bar{\Delta}_{21}(x, \rho)e^{\rho i(1-x)}e^{\rho i(1-y)} + \bar{\Delta}_{22}(x, \rho)e^{\rho i(1-x)}e^{\rho i y})$$

for  $\rho$  on  $S'_1$ .

We see immediately that for  $\rho$  on  $S'_1$

$$2\rho(G(x, y; \rho^2) - \bar{G}(x, y; \rho^2)) = \{ e^{\rho i(x-y)}m_1(x, y, \rho)/\rho; e^{-\rho i(x-y)}m_2(x, y; \rho)/\rho \} \\ + e^{\rho i x}m_3(x, y, \rho) + e^{\rho i(1-x)}m_4(x, y, \rho).$$

Thus, by Lemmas III and V' of our preceding paper,

$$\int_{\gamma_1} \left( 1 - \frac{\rho^2}{R^2} \right)^l 2\rho(G - \bar{G})d\rho, \quad l \geq 0,$$

is uniformly bounded,  $0 < a \leq x \leq b < 1$ ,  $0 \leq y \leq 1$ , for all  $\gamma_1$  on  $S'_1$ .

To discuss the integral

$$\int_a^x \int_{\gamma_1} \left( 1 - \frac{\rho^2}{R^2} \right)^l 2\rho(G - \bar{G})d\rho dy, \quad l \geq 0,$$

we observe that

$$\int_a^x e^{\rho i(1-y)} \left( 1 + \frac{B_1(y)}{\rho i} + \frac{E_1}{\rho^2} \right) dy = \frac{e^{\rho i(1-a)} - e^{\rho i(1-x)}}{\rho i} \\ + \frac{B_1(x)e^{\rho i(1-x)} - B_1(a)e^{\rho i(1-a)}}{\rho^2} \\ + \int_a^x e^{\rho i(1-y)} \frac{E_1 - B_1'(y)}{\rho^2} dy \\ = \frac{e^{\rho i(1-a)} - e^{\rho i(1-x)}}{\rho i} + \frac{m(x, \alpha, \rho)}{\rho^2}, \\ \int_a^x e^{\rho i y} \left( 1 - \frac{B_1(y)}{\rho i} + \frac{E_2}{\rho^2} \right) dy = \frac{e^{\rho i x} - e^{\rho i a}}{\rho i} + \frac{m(x, \alpha, \rho)}{\rho^2}.$$

Thus we have

$$\begin{aligned} \int_a^x 2\rho(G - \bar{G})dy &= \int_a^x \{e^{\rho i(x-y)}m_1(x, y, \rho)/\rho; e^{-\rho i(x-y)}m_2(x, y, \rho)/\rho\}dy \\ &+ \frac{1}{B(e^{2\rho i} + \mu\rho e^{\rho i} - 1)} \left( e^{\rho i x} \int_a^x (e^{\rho i(1-y)}[\Delta_{11}] \right. \\ &- \bar{\Delta}_{11}(x, \rho)e^{\rho i(1-y)})dy + \dots + e^{\rho i(1-x)} \int_a^x (e^{\rho i y}[\Delta_{22}] \\ &- \bar{\Delta}_{22}(x, \rho)e^{\rho i y})dy \Big), \end{aligned}$$

where the coefficients of  $e^{\rho i x}$ ,  $e^{\rho i(1-x)}$  are of the form  $m_3(x, \alpha, \rho)/\rho$ ,  $m_4(x, \alpha, \rho)/\rho$  respectively; we compute, for instance,

$$\begin{aligned} \int_a^x (e^{\rho i(1-y)}[\Delta_{11}] - \bar{\Delta}_{11}(x, \rho)e^{\rho i(1-y)})dy &= (\Delta_{11}(x, \rho) - \bar{\Delta}_{11}(x, \rho)) \frac{e^{\rho i(1-x)} - e^{\rho i(1-x)}}{\rho i} \\ &+ \frac{\Delta_{11}(x, \rho)m_1(x, \alpha, \rho)}{\rho^2} + \frac{\bar{\Delta}_{11}(x, \rho)m_2(x, \alpha, \rho)}{\rho^2} = m(x, \alpha, \rho)/\rho. \end{aligned}$$

Hence we find

$$\begin{aligned} \int_a^x 2\rho(G - \bar{G})dy &= \int_a^x \{e^{\rho i(x-y)}m_1(x, y, \rho)/\rho; e^{-\rho i(x-y)}m_2(x, y, \rho)/\rho\}dy \\ &+ e^{\rho i x}m_3(x, \alpha, \rho)/\rho + e^{\rho i(1-x)}m_4(x, \alpha, \rho). \end{aligned}$$

Lemmas V' and VI' of our antecedent paper show that

$$\int_a^x \int_{\gamma_1} \left(1 - \frac{\rho^8}{R^8}\right)^l 2\rho(G(x, y; \rho^2) - \bar{G}(x, y; \rho^2))d\rho dy \rightarrow 0, \quad l \geq 0,$$

as  $R \rightarrow \infty$ , uniformly,  $0 < a \leq x \leq b < 1$ .

The reasoning then follows the usual channels until we have

$$\lim_{R \rightarrow \infty} \int_0^1 f(y) \int_{\gamma_1} \left(1 - \frac{\rho^8}{R^8}\right)^l 2\rho(G(x, y; \rho^2) - \bar{G}(x, y; \rho^2))d\rho dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ ; and

$$\lim_{\lambda \rightarrow \infty} \int_0^1 f(y) \int_{C_\lambda} \left(1 - \frac{\lambda^4}{\Lambda_\lambda^4}\right)^l (G(x, y; \lambda) - \bar{G}(x, y; \lambda))d\lambda dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ . The proof is thus completed.

Because of the result just obtained, it is of interest to study the differential system

$$\begin{aligned} u'' + \lambda u &= 0, & \lambda &= \rho^2, & 0 \leq x \leq 1, \\ u'(0) + Au'(1) + Bu(1) &= 0, & B &\neq 0, \\ u(0) - Au(1) &= 0. \end{aligned}$$

We find that there are three cases to consider according as  $A = -1$ ,  $A = +1$ ,  $A^2 - 1 \neq 0$ ; we shall call them Cases I, II, III, and take them up in order.

In Case I the differential system and its adjoint are respectively

$$\begin{aligned} u'' + \rho^2 u &= 0, & v'' + \rho^2 v &= 0, \\ u'(0) - u'(1) + Bu(1) &= 0, & v'(0) + v'(1) - Bv(0) &= 0, \\ u(0) + u(1) &= 0, & v(0) - v(1) &= 0. \end{aligned}$$

The characteristic values of  $\rho$  are found from the equation  $e^{\rho i} - e^{-\rho i} = 0$ ; in fact, they are all simple and are given by  $\rho = k\pi$ ,  $k = \pm 1, \pm 2, \dots$ . We need consider only positive values of  $k$ . The corresponding solutions of the differential system are then

$$\begin{aligned} u_1 &= \sqrt{2} \sin \pi x + \frac{2\sqrt{2}}{B} \pi \cos \pi x, & v_1 &= \sqrt{2} \sin \pi x, \\ u_2 &= \sqrt{2} \sin 2\pi x, & v_2 &= \sqrt{2} \sin 2\pi x + \frac{2\sqrt{2}}{B} 2\pi \cos 2\pi x, \\ u_{2m} &= \sqrt{2} \sin 2m\pi x, & v_{2m} &= \sqrt{2} \sin 2m\pi x + \frac{2\sqrt{2}}{B} 2m\pi \cos 2m\pi x, \\ &\dots & \dots & \\ u_{2m+1} &= \sqrt{2} \sin (2m+1)\pi x + \frac{2\sqrt{2}}{B} (2m+1)\pi \cos (2m+1)\pi x, \\ & & v_{2m+1} &= \sqrt{2} \sin (2m+1)\pi x, \end{aligned}$$

as can be verified by direct substitution. For these solutions we find

$$\int_0^1 u_i v_k dx = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}.$$

The expansion problem is therefore that of representing an arbitrary summable function in terms of the infinite series

$$\sum_{k=1}^{\infty} a_k u_k, \quad a_k = \int_0^1 f v_k dx.$$

The sum of the first  $N$  terms of this series can be written

$$\sum_{k=1}^{k=N} a_k u_k = \sum_{k=1}^{k=N} a_k' \sqrt{2} \sin k\pi x + \frac{4}{B} \sum_{k=1}^{k=N} \frac{d}{dx} (a_k'' \cos k\pi x + b_k'' \sin k\pi x),$$

where

$$\begin{aligned} a_k' &= \sqrt{2} \int_0^1 f(y) \sin k\pi y dy, \\ a_{2m}'' &= - \int_0^1 f(y) \cos 2m\pi y dy, \\ a_{2m+1}'' &= 0, \\ b_{2m}'' &= 0, \\ b_{2m+1}'' &= \int_0^1 f(y) \sin (2m+1)\pi y dy. \end{aligned}$$

If  $F_1(x)$  is defined for the interval  $(0, 2)$  by the equations

$$\begin{aligned} F_1(x) &= 0, \quad 0 \leq x \leq 1, \\ F_1(x) &= \frac{f(x-1) + f(2-x)}{2}, \quad 1 \leq x \leq 2, \end{aligned}$$

its expansion in terms of Fourier series on the interval  $(0, 2)$  is given by

$$A_0 + \sum_{k=1}^{k=\infty} (A_k \cos k\pi x + B_k \sin k\pi x),$$

where

$$\begin{aligned} A_0 &= \frac{1}{2} \int_0^2 F_1(y) dy, \\ A_k &= \int_0^2 F_1(y) \cos k\pi y dy = \int_0^1 \frac{f(y) + f(1-y)}{2} \cos k\pi(y+1) dy = a_k'', \\ B_k &= \int_0^2 F_1(y) \sin k\pi y dy = \int_0^1 \frac{f(y) + f(1-y)}{2} \sin k\pi(y+1) dy = b_k'', \end{aligned}$$

by a series of obvious manipulations. In other words, the expression

$$0 + \sum_{k=1}^{k=N} \frac{d}{dx} (a_k'' \cos k\pi x + b_k'' \sin k\pi x)$$

is the sum of the first  $2N+1$  terms of the term-by-term derived series of the Fourier series for  $F_1(x)$ , a function identically zero,  $0 \leq x < 1$ . We recall at this point some of the theorems concerning the derived series of Fourier

series. In particular the Cesàro mean of order  $l > 0$  for the present series converges uniformly to zero,  $0 < a \leq x \leq b < 1$ .<sup>\*</sup> The term  $\Sigma a_k \sqrt{2} \sin k\pi x$  is the ordinary sine series on  $(0, 1)$ .

Thus from Theorem VI of our preceding paper and Theorem IX of the present one we have

**THEOREM X.** *If  $G(x, y; \lambda)$  is the Green's function for an irregular differential system of the second order of type 1, Case I, and if  $f(x)$  is summable on  $(0, 1)$ , then the expression*

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l G(x, y; \lambda) d\lambda dy, \quad l \geq 0,$$

*is equivalent on any interval  $(a, b)$  completely interior to  $(0, 1)$  to a sum of means of order  $l$  formed from the sine series on  $(0, 1)$  for  $f(x)$  and from the derived series of the Fourier series on  $(0, 2)$  for  $F_1(x)$ , where  $F_1(x)$  is the function defined above. In consequence,*

$$\lim_{p \rightarrow \infty} \frac{1}{2\pi i} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l G(x, y; \lambda) d\lambda dy = f(x), \quad l > 0,$$

*almost everywhere,  $0 < x < 1$ ; the convergence is uniform on  $(a, b)$  if  $f(x)$  is continuous on  $(0, 1)$ .*

This theorem is stronger in the case  $M=1$ ,  $A=-1$ , than Theorem V; it has also the advantage of revealing clearly the precise nature of the irregularity in the expansion problem.

In Case II,  $A=+1$ , we obtain entirely similar results. The differential system can be solved and the formal series set up as before. It is found that the expansion of an arbitrary summable function  $f(x)$  is representable as the sum of the sine series on  $(0, 1)$  for  $f(x)$  and the term-by-term derived series of the Fourier series on  $(0, 2)$  for  $F_2(x)$ , where

$$\begin{aligned} F_2(x) &= 0, & 0 \leq x \leq 1, \\ F_2(x) &= \frac{f(2-x) - f(x-1)}{2}, & 1 \leq x \leq 2. \end{aligned}$$

It is therefore possible to state the following theorem.

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<sup>\*</sup> W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 13 (1914), pp. 13-28; also § VI of our preceding paper.

THEOREM XI. If  $G(x, y; \lambda)$  is the Green's function for an irregular differential system of the second order of type 1, Case II, then the expression

$$\frac{1}{2\pi i} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l G(x, y; \lambda) d\lambda dy, \quad l \geq 0,$$

formed for any summable function  $f(x)$ , is equivalent on any interval  $(a, b)$  completely interior to  $(0, 1)$  to a sum of means of order  $l$  formed from the sine series for  $f(x)$  on  $(0, 1)$  and from the derived series of the Fourier series for  $F_2(x)$  on  $(0, 2)$ , where  $F_2(x)$  is the function defined above. In consequence

$$\lim_{l \rightarrow \infty} \frac{1}{2\pi i} \int_0^1 f(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l G(x, y; \lambda) d\lambda dy = f(x), \quad l > 0,$$

almost everywhere,  $0 < x < 1$ ; the convergence is uniform on  $(a, b)$  if  $f(x)$  is continuous on  $(0, 1)$ .

To discuss Case III,  $A^2 - 1 \neq 0$ , we first prove

LEMMA IV. If  $\rho = r + is$ ,  $0 < C < |\mu|$ , then

$$\left| \frac{1}{e^{2\rho s} + \mu \rho e^{\rho s} - 1} \right| \leq K e^s / r, \quad 0 \leq s \leq \log Cr, \quad r \geq 0,$$

for  $|\rho|$  sufficiently large; and

$$\left| \frac{1}{1 + \mu \rho e^{-\rho s} - e^{-2\rho s}} \right| \leq K e^{-s} / r, \quad -\log Cr \leq s \leq 0, \quad r \geq 0,$$

for  $|\rho|$  sufficiently large.

We take up the first inequality only, the other being treated similarly. We have

$$e^{2\rho s} + \mu \rho e^{\rho s} - 1 = \mu \rho e^{\rho s} [1] - 1.$$

Hence

$$\begin{aligned} |e^{2\rho s} + \mu \rho e^{\rho s} - 1| &= |\mu \rho e^{\rho s} [1] - 1| \\ &= r e^{-s} \left| \mu \left( 1 + \frac{is}{r} \right) (\cos r + i \sin r) [1] - \frac{e^s}{r} \right| \\ &\geq r e^{-s} \left( |\mu| \left| 1 + \frac{is}{r} \right| |[1]| - C \right) > \frac{r e^{-s}}{K}, \quad 0 \leq s \leq \log Cr, \quad r \geq 0, \end{aligned}$$

for  $|\rho|$  sufficiently large, since the term in the last parenthesis has the positive limit  $|\mu| - C$  as  $|\rho| \rightarrow \infty$ , uniformly for the range of  $s$  considered.

We can now prove

**THEOREM XII.** *If  $G(x, y; \lambda)$  is the Green's function for an irregular differential system of the second order of type 1, Case III;  $\bar{G}(x, y; \lambda)$  is the Green's function for the Fourier differential system of the second order*

$$\begin{aligned}u'' + \lambda u &= 0, \\u'(0) - u'(1) &= 0, \\u(0) - u(1) &= 0;\end{aligned}$$

and  $\varphi(x)$  is of bounded variation on  $(0, 1)$ ; then

$$\lim_{l \rightarrow \infty} \frac{1}{2\pi i} \int_0^1 \varphi(y) \int_C \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l (G(x, y; \lambda) - \bar{G}(x, y; \lambda)) d\lambda dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ . Thus

$$\lim_{l \rightarrow \infty} \frac{1}{2\pi i} \int_0^1 \varphi(y) \int_C \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l G(x, y; \lambda) d\lambda dy = \frac{\varphi(x+0) + \varphi(x-0)}{2}, \quad l \geq 0,$$

$0 < x < 1$ ; the convergence is uniform on  $(a, b)$  if  $\varphi(x)$  is continuous on  $(0, 1)$ .

It suffices to prove the theorem for a monotone function  $\varphi(x)$  and, by Theorem IX, the Green's function for the system

$$\begin{aligned}u'' + \lambda u &= 0, \\u'(0) + Au'(1) + Bu(1) &= 0, \\u(0) - Au(1) &= 0, \quad B(1 - A^2) \neq 0.\end{aligned}$$

We apply the second law of the mean for integrals to the expressions

$$\begin{aligned}\int_0^1 \varphi(y) e^{\rho i(1-y)} dy &= \varphi(+0) \int_0^{t_1} e^{\rho i(1-y)} dy + \varphi(1-0) \int_{t_1}^1 e^{\rho i(1-y)} dy = \frac{m(\rho)}{\rho}, \\ \int_0^1 \varphi(y) e^{\rho i y} dy &= \varphi(+0) \int_0^{t_2} e^{\rho i y} dy + \varphi(1-0) \int_{t_2}^1 e^{\rho i y} dy = \frac{m(\rho)}{\rho}.\end{aligned}$$

On substituting these results in the expression for  $G$  given in Theorem IX, we find

$$\begin{aligned}\int_0^1 \varphi(y) 2\rho \left(1 - \frac{\rho^8}{R^8}\right)^l G(x, y; \rho^2) dy &= \int_0^1 \varphi(y) \{ -ie^{\rho i(x-y)}; \\ &\quad -ie^{-\rho i(x-y)} \} \left(1 - \frac{\rho^8}{R^8}\right)^l dy + \frac{e^{\rho i x} m_1(\rho) + e^{\rho i(1-x)} m_2(\rho)}{e^{2\rho i} + \mu \rho e^{\rho i} - 1}, \quad l \geq 0.\end{aligned}$$

We now show that

$$\lim_{R \rightarrow \infty} \int_{\gamma_1} \frac{e^{\rho i x} m_1(\rho) + e^{\rho i(1-x)} m_2(\rho)}{e^{2\rho i} + \mu \rho e^{\rho i} - 1} d\rho = 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ . To do this we write  $\rho = Re^{i\theta}$ ,  $0 \leq \theta \leq \pi/2$ , on  $\gamma_1$ , and then investigate the integrals for  $0 \leq \theta \leq \theta_1$ ,  $\theta_1 \leq \theta \leq \pi/4$ ,  $\pi/4 \leq \theta \leq \pi/2$ , where  $\theta_1$  satisfies the equation  $R \sin \theta_1 = \log CR \sin \theta_1$ . That  $\theta_1$  exists and is unique is seen very readily; it is the argument of the point of intersection of  $\gamma_1$  and the curve  $Cr = e^x$ . Then we have, if  $\delta > 0$  is the lesser of  $a$ ,  $1-b$ , and if  $|m_1| \leq M/2$ ,  $|m_2| \leq M/2$ , because of Lemma IV,

$$\begin{aligned} \left| \int_0^{\theta_1} \frac{e^{\rho i x} m_1 + e^{\rho i(1-x)} m_2}{e^{2\rho i} + \mu \rho e^{\rho i} - 1} Re^{i\theta} d\theta \right| &\leq MK \int_0^{\theta_1} \frac{e^{R(1-\delta)\sin \theta}}{\cos \theta} d\theta \\ &\leq \frac{MK}{\cos^2 \theta_1} \int_0^{\theta_1} e^{R(1-\delta)\sin \theta} \cos \theta d\theta = \frac{MK}{\cos^2 \theta_1} \frac{e^{R(1-\delta)} - 1}{R(1-\delta)} \\ &= \frac{MK}{\cos^2 \theta_1} \frac{(CR \cos \theta_1)^{1-\delta} - 1}{R(1-\delta)} \rightarrow 0, \quad \theta_1 \rightarrow 0, \end{aligned}$$

as  $R \rightarrow \infty$ . Again, by Lemma II,

$$\begin{aligned} \left| \int_{\theta_1}^{\pi/4} \frac{e^{\rho i x} m_1 + e^{\rho i(1-x)} m_2}{e^{2\rho i} + \mu \rho e^{\rho i} - 1} Re^{i\theta} d\theta \right| &\leq MK \int_{\theta_1}^{\pi/4} e^{-\delta R \sin \theta} R d\theta \\ &\leq \frac{MK}{\cos(\pi/4)} \int_{\theta_1}^{\pi/4} e^{-\delta R \sin \theta} R \cos \theta d\theta = \frac{MK}{\delta \cos(\pi/4)} (e^{-R \delta \sin(\pi/4)} - e^{-\delta R \sin \theta_1}) \\ &= \frac{MK}{\delta \cos(\pi/4)} (e^{-R \delta \sin(\pi/4)} - (CR \cos \theta_1)^{-\delta}) \rightarrow 0 \end{aligned}$$

as  $R \rightarrow \infty$ . Finally

$$\left| \int_{\pi/4}^{\pi/2} \frac{e^{\rho i x} m_1 + e^{\rho i(1-x)} m_2}{e^{2\rho i} + \mu \rho e^{\rho i} - 1} Re^{i\theta} d\theta \right| \leq MK \int_{\pi/4}^{\pi/2} Re^{-\delta R \sin(\pi/4)} d\theta \rightarrow 0$$

as  $R \rightarrow \infty$ . We have established the desired result.

It follows at once that

$$\lim_{R \rightarrow \infty} \int_0^1 \varphi(y) \int_{\gamma_1} \left( 1 - \frac{\rho^8}{R^8} \right)^l (2\rho G - \{ -ie^{\rho i(x-y)} ; -ie^{-\rho i(x-y)} \}) d\rho dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ . We know that we can replace  $G$  by  $\bar{G}$ , the Green's function for the differential system

$$u'' + \lambda u = 0,$$

$$u'(0) - u'(1) = 0,$$

$$u(0) - u(1) = 0,$$

in this expression. Thus

$$\lim_{R \rightarrow \infty} \int_0^1 \varphi(y) \int_{\gamma_1} \left(1 - \frac{\rho^8}{R^8}\right)^l 2\rho(G - \bar{G})d\rho dy = 0, \quad l \geq 0,$$

and

$$\lim_{\lambda \rightarrow \infty} \int_0^1 \varphi(y) \int_{C_r} \left(1 - \frac{\lambda^4}{\Lambda^4}\right)^l (G(x, y; \lambda) - \bar{G}(x, y; \lambda))d\lambda dy = 0, \quad l \geq 0,$$

uniformly,  $0 < a \leq x \leq b < 1$ . The remainder of the theorem follows at once.

The differential system

$$u'' + \lambda u = 0, \quad 0 \leq x \leq 1,$$

$$u'(0) + Au'(1) + Bu(1) = 0,$$

$$u(0) - Au(1) = 0, \quad B \neq 0,$$

was so considered in Cases I and II that knowledge concerning the behavior at  $x=0$  and at  $x=1$  of the expansions associated with them was contained in the theorems proved; these two points are obviously points at which the irregularity of the differential system renders the expansions especially peculiar. In Case III we must study the expansions at these points separately. We take the characteristic equation in the form  $\sin \rho = \bar{\mu}\rho$  instead of  $e^{\rho i} + \mu\rho - e^{-\rho i} = 0$ . For large  $|\rho|$  the characteristic values corresponding to roots of this equation are all simple. We denote them by  $\rho_K, \rho_{K+1}, \rho_{K+2}, \dots$ , where  $|\rho_{k+1}| \geq |\rho_k|$ ,  $k=K, K+1, \dots$ . Since  $A^2 - 1 \neq 0$ , the functions

$$U_k = \sin \rho_k x + A \sin \rho_k (1-x)$$

and

$$V_k = A \sin \rho_k x - \sin \rho_k (1-x) \quad (k=K, K+1, \dots),$$

satisfy the differential system and its adjoint for  $\rho = \rho_k$ .

Then

$$\begin{aligned}\int_0^1 U_k V_k dy &= 0, \quad i \neq k, \quad i, k \geq K, \\ \int_0^1 U_k V_k dy &= A \int_0^1 (\sin^2 \rho_k y - \sin^2 \rho_k (1-y)) dy \\ &\quad + (A^2 - 1) \int_0^1 \sin \rho_k y \sin \rho_k (1-y) dy \\ &= \frac{A^2 - 1}{2} \int_0^1 (\cos \rho_k (2y - 1) - \cos \rho_k) dy \\ &= \frac{1 - A^2}{2} \left( \cos \rho_k - \frac{\sin \rho_k}{\rho_k} \right) = \frac{1 - A^2}{2} (\cos \rho_k - \bar{\mu}).\end{aligned}$$

The expansion of an arbitrary summable function thus takes the form

$$\sum_{k=1}^{k=K} \int_0^1 f(y) R_k(x, y) dy + \sum_{k=K}^{k=\infty} \frac{2}{1 - A^2} \frac{U_k(x)}{\cos \rho_k - \bar{\mu}} \int_0^1 f V_k dy.$$

In the case that  $A=0$ , the boundary condition  $u(0)=0$  shows us that for  $x=0$  this expansion converges to zero; hence we consider the case  $x=0$  only when  $A \neq 0$ . We shall discuss the convergence at  $x=0$ ,  $x=1$  of the above series for a function  $f(x)$  continuous with its first three derivatives on  $(0, 1)$ . It is unnecessary to treat the first  $K-1$  terms for our purpose. We have at once

$$\begin{aligned}\int_0^1 f V_k dy &= -f(y)(A \cos \rho_k y + \cos \rho_k (1-y))/\rho_k \Big|_{y=0}^{y=1} \\ &\quad + f'(y)(A \sin \rho_k y - \sin \rho_k (1-y))/\rho_k^2 \Big|_{y=0}^{y=1} \\ &\quad + f''(y)(A \cos \rho_k y + \cos \rho_k (1-y))/\rho_k^3 \Big|_{y=0}^{y=1} \\ &\quad - \frac{1}{\rho_k^3} \int_0^1 f'''(y)(A \cos \rho_k y + \cos \rho_k (1-y)) dy.\end{aligned}$$

The first two terms combine as  $(\alpha \cos \rho_k + \beta)/\rho_k$  where  $\alpha, \beta$  are constants;  $\alpha$ , in particular, is the expression  $f(0) - Af(1)$ . Since we have

$$\begin{aligned} \left| \frac{\cos \rho_k \xi}{\rho_k} \right| &= \frac{1}{|\rho_k|} (\cosh^2 s_k \xi \cos^2 r_k \xi + \sinh^2 s_k \xi \sin^2 r_k \xi)^{1/2} \\ &= \frac{1}{|\rho_k|} (\cosh^2 s_k \xi - \sin^2 r_k \xi)^{1/2} \leq \cosh s_k \xi / |\rho_k| \\ &\leq \cosh s_k / |\rho_k| \leq Q \end{aligned}$$

for  $0 \leq \xi \leq 1$ ,  $\rho_k = r_k + is_k$ ,  $k = K, K+1, \dots$ , by the results of Theorem IV, we can write the sum of the last two terms as  $m_k/\rho_k^2$  where  $m_k$  is bounded,  $k = K, K+1, \dots$ . We have, then, to consider the series

$$\sum_{k=K}^{\infty} \frac{2}{1-A^2} \frac{U_k(x)}{\rho_k} \left( \frac{\alpha \cos \rho_k + \beta}{\cos \rho_k - \bar{\mu}} + \frac{m_k \rho_k}{\rho_k^2 (\cos \rho_k - \bar{\mu})} \right)$$

at  $x=0$ ,  $x=1$ . Since  $\sin \rho_k = \bar{\mu} \rho_k$  these series are

$$\begin{aligned} \frac{2A\bar{\mu}}{1-A^2} \sum_{k=K}^{\infty} \left( \frac{\alpha \cos \rho_k + \beta}{\cos \rho_k - \bar{\mu}} + \frac{m_k \rho_k}{\rho_k^2 (\cos \rho_k - \bar{\mu})} \right), \\ \frac{2\bar{\mu}}{1-A^2} \sum_{k=K}^{\infty} \left( \frac{\alpha \cos \rho_k + \beta}{\cos \rho_k - \bar{\mu}} + \frac{m_k \rho_k}{\rho_k^2 (\cos \rho_k - \bar{\mu})} \right). \end{aligned}$$

Since  $\cos \rho_k = \pm \sqrt{1 - \sin^2 \rho_k} = \pm \sqrt{1 - \bar{\mu}^2 \rho_k^2}$  we have

$$\liminf_{k \rightarrow \infty} \left| \frac{\cos \rho_k}{\cos \rho_k - \bar{\mu}} \right| > 0,$$

and the series  $\sum \cos \rho_k / (\cos \rho_k - \bar{\mu})$  is divergent. Similarly, we find

$$\left| \frac{m_k \rho_k}{\rho_k^2 (\cos \rho_k - \bar{\mu})} \right| \leq \frac{M}{|\rho_k|^2} \quad (k = K, K+1, \dots).$$

Since by Theorem IV  $\sum 1/|\rho_k|^2$  is comparable to  $\sum 1/k^2$ , the series  $\sum m_k \rho_k / \rho_k^2 (\cos \rho_k - \bar{\mu})$  is convergent. Lastly we show that  $\sum 1/(\cos \rho_k - \bar{\mu})$  converges. If  $C_l$  is a simple closed contour on  $\Sigma'$  surrounding  $\rho_K, \dots, \rho_{K+l}$ , then

$$\sum_{k=K}^{K+l} \frac{1}{\cos \rho_k - \bar{\mu}} = \frac{1}{2\pi i} \int_{C_l} \frac{d\rho}{\sin \rho - \bar{\mu} \rho},$$

by the theory of residues. We shall take  $C_l$  as being the contour made up of two concentric semi-circles on  $\Sigma'$ , namely  $\Gamma_0$  and  $\gamma_1 + \gamma_{IV}$ , joined by segments of the imaginary axis which we shall call  $\gamma_1$  and  $\gamma_4$  respectively. We find

$$\int_{\gamma_1} \frac{d\rho}{\sin \rho - \bar{\mu} \rho} = - \int_{\gamma_4} \frac{d\rho}{\sin \rho - \bar{\mu} \rho}$$

by replacing  $\rho$  by  $-\rho$  in either integral. By work like that of Theorem XII we have

$$\int_{\gamma_1} \frac{d\rho}{\sin \rho - \bar{\mu}\rho} = \int_{\gamma_1} \frac{2ie^{\rho i} d\rho}{e^{2\rho i} + \bar{\mu}\rho e^{\rho i} - 1} \rightarrow 0, \quad \int_{\gamma_{IV}} \frac{d\rho}{\sin \rho - \bar{\mu}\rho} \rightarrow 0,$$

as  $R \rightarrow \infty$ . By the use of these facts we find

$$\lim_{l \rightarrow \infty} \sum_{k=K}^{k=K+l} \frac{1}{\cos \rho_k - \bar{\mu}} = \frac{1}{2\pi i} \int_{\gamma_0} \frac{d\rho}{\sin \rho - \bar{\mu}\rho},$$

and the desired result is proved.

In short, if  $\alpha = f(0) - Af(1)$  is different from zero, the expansions for  $f(x)$ , continuous together with its first three derivatives on  $(0, 1)$ , in Case III diverge at  $x=0$  and  $x=1$ , except when  $A=0$ ,  $x=0$ ; if  $\alpha=0$  they converge. In Case III, therefore, the expansions present special irregularities at the end points of the interval of definition.

COLUMBIA UNIVERSITY,  
NEW YORK, N.Y.

# SOME PROBLEMS IN THE THEORY OF INTERPOLATION BY STURM-LIOUVILLE FUNCTIONS\*

BY  
CAREY M. JENSEN

A problem which receives a large share of attention in modern analysis consists in the determination of the properties of linear combinations of orthogonal functions, particularly with reference to the possibility of obtaining, by these combinations, approximate representations of certain classes of functions. Let  $\varphi_0(x), \varphi_1(x), \dots, \varphi_p(x)$  denote a sequence of functions orthogonal in the interval  $a \leq x \leq b$ , and let  $f(x)$  represent an arbitrary function. The linear combination

$$(1) \quad \alpha_0 \varphi_0(x) + \alpha_1 \varphi_1(x) + \dots + \alpha_p \varphi_p(x),$$

where the coefficients are defined thus:

$$\alpha_n = \frac{\int_a^b f(x) \varphi_n(x) dx}{\int_a^b \varphi_n^2(x) dx} \quad (n=0, 1, \dots, p),$$

is studied with reference to the question of its convergence toward  $f(x)$ , as  $p$  is allowed to increase without limit. The classical example of such series is the Fourier cosine series.

The value of the coefficient  $\alpha_n$ , as defined above, depends upon the behavior of  $f(x)$  everywhere throughout the interval  $a \leq x \leq b$ . Another class of problems arises if the coefficient is defined so that its value shall depend upon the values of  $f(x)$  only at discrete points of the interval. In particular, let the interval  $a \leq x \leq b$  be subdivided into  $p$  equal parts by the points  $x_0 = a, x_1, x_2, \dots, x_p = b$ , and consider the sum

$$(2) \quad \alpha_{0p} \varphi_0(x) + \alpha_{1p} \varphi_1(x) + \dots + \alpha_{pp} \varphi_p(x),$$

where

$$\alpha_{np} = \frac{\sum_{k=0}^p f(x_k) \varphi_n(x_k)}{\sum_{k=0}^p \varphi_n^2(x_k)} \quad (n=0, 1, \dots, p),$$

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the symbol  $\sum'$  being used in the following sense:

$$\sum_{k=0}^p y_k = \frac{1}{2} y_0 + \sum_{k=1}^{p-1} y_k + \frac{1}{2} y_p.$$

The expression given in (2) will be referred to as the interpolating formula, of order  $p$ , for  $f(x)$  with respect to the orthogonal system  $\varphi_0(x), \varphi_1(x), \dots$ , in the sense that it is a formula of approximation determined by the values of  $f(x)$  at a finite number of points, not that it necessarily takes on the values of  $f(x)$  at these points. The classical example of such a formula is found in the cosine interpolating formula, which, as we shall presently indicate, may be regarded as a special case of the ordinary formula for trigonometric interpolation. Investigations into the properties of the latter, by de la Vallée Poussin,\* Faber,† Jackson,‡ and others, have yielded results which are noteworthy because of their close parallelism, both in substance and mode of attainment, to those obtaining in the case of Fourier series. Further problems suggested by a consideration of (2) are quite similar to those studied in connection with (1), but in the case of (2) the solutions have not, in general, been so extensively worked out.

The particular orthogonal function system with which we shall deal in this paper is formed by the characteristic functions of the so-called Sturm-Liouville differential system

$$\begin{aligned} (I) \quad & u''(x) + [\rho^2 - \lambda(x)]u(x) = 0, \\ & u'(0) - hu(0) = 0, \\ & u'(\pi) + Hu(\pi) = 0, \end{aligned}$$

where  $h$  and  $H$  are real, but unrestricted as to sign, and  $\lambda(x)$  is for the present merely defined and continuous in the interval  $0 \leq x \leq \pi$ . Let the solutions of this system corresponding to the characteristic numbers  $\rho_0^2, \rho_1^2, \dots$ , arranged in order of magnitude, be denoted by  $u_0(x), u_1(x), \dots$ , and consider the sum

$$\sum_p [f(x)] = \alpha_{0p} u_0(x) + \alpha_{1p} u_1(x) + \dots + \alpha_{pp} u_p(x),$$

\* C. de la Vallée Poussin, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bulletins de l'Académie Royale de Belgique, Classe des Sciences, 1908, pp. 319-403.

† Faber, *Über stetige Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 372-441; pp. 417-424.

‡ D. Jackson, *On the accuracy of trigonometric interpolation*, these Transactions, vol. 14 (1913), pp. 453-461, pp. 453-456.

where

$$\alpha_{np} = \frac{\sum_{k=0}^p f(x_k) u_n(x_k)}{\sum_{k=0}^p u_n^2(x_k)} \quad (n=0, 1, \dots, p).$$

The expression  $\sum_p[f(x)]$ , which will be referred to as the Sturm-Liouville interpolating formula for  $f(x)$ , constitutes the subject for investigation.

In the discussion, some reference will be made to sums closely allied, in one way or another, with  $\sum_p[f(x)]$ . These are the following:

(a) the partial sum of the Sturm-Liouville series,

$$\sigma_p[f(x)] = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \dots + \alpha_p u_p(x),$$

where

$$\alpha_n = \frac{\int_0^\pi f(x) u_n(x) dx}{\int_0^\pi u_n^2(x) dx};$$

(b) the cosine interpolation formula, of order  $p$ ,

$$T_p[f(x)] = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_p \cos px,$$

where

$$a_{np} = \frac{\sum_{k=0}^p f(x_k) \cos nx_k}{\sum_{k=0}^p \cos^2 nx_k};$$

(c) the partial sum of the Fourier cosine series,

$$t_p[f(x)] = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_p \cos px,$$

where

$$a_n = \frac{\int_0^\pi f(x) \cos nx dx}{\int_0^\pi \cos^2 nx dx}.$$

It should be noticed here that if we specialize the Sturm-Liouville differential system by setting  $h = H = 0$ ,  $\lambda(x) \equiv 0$ , the resulting characteristic solutions are precisely the cosine functions. Furthermore, it can be readily shown that, if  $f(x)$  is defined outside the interval  $0 \leq x \leq \pi$  so as to make it

an even periodic function of period  $2\pi$ , then the ordinary formula for trigonometric interpolation, using an even number ( $2p$ ) of interpolating points\* evenly distributed throughout the interval  $0 \leq x \leq 2\pi$ , reduces precisely to the cosine formula  $T_p[f(x)]$ . An analogous relation exists between  $t_p[f(x)]$  and the partial sum of the ordinary Fourier series.

A brief outline of the topics to be treated is as follows. In the first section are listed a number of facts concerning the nature of the characteristic numbers and solutions of the Sturm-Liouville differential system. In Section 2 there is outlined a proof of the convergence of  $\sum_p[f(x)]$  to  $f(x)$ , provided  $f(x)$  satisfies suitable conditions, followed in Section 3 by a detailed proof of the so-called "equivalence" theorem. In the last section there is outlined briefly the method by which we establish another theorem, concerning the rapidity of convergence of  $\sum_p[\varphi_p(x)]$  to  $\varphi_p(x)$ , where  $\varphi_p(x)$  is itself a Sturm-Liouville sum. The statement of a corollary, relative to the rapidity of convergence of  $\sum_p[f(x)]$  to  $f(x)$ , when the latter satisfies a Lipschitz condition, concludes the paper. The analysis is rather laborious, especially in the last section, and, to keep the paper from running to inordinate length, the exposition has been much condensed. It is believed, however, that the indications are sufficient to enable the reader to supply the missing details with reasonable directness (except possibly in the case of Theorem IV, which is merely stated without proof, and of which no further use is made). Copies of a more complete version in manuscript are on file in the library of the University of Minnesota and in the library of the Society.

1. **Preliminary statements.** (a) The solutions of the system

$$\begin{aligned}u''(x) + [\rho^2 - \lambda(x)]u(x) &= 0, \\ u'(0) - hu(0) &= 0,\end{aligned}$$

cannot be essentially complex, provided that  $\rho^2$ ,  $h$ , and  $\lambda(x)$  are real.

(b) There are infinitely many real values of  $\rho^2$  for which the system (I) is compatible; they have no cluster point in the finite plane, and only a finite number of them can be negative. To each of these values of  $\rho^2$  corresponds a solution  $u(x)$  uniquely determined except for a multiplicative constant.

(c) If the index  $n$  be chosen such that  $\rho_0^2 < \rho_1^2 < \rho_2^2 \cdots$ , then the characteristic function  $u_n(x)$  corresponding to  $\rho_n$  will possess precisely  $n$  zeros in the interval†  $0 \leq x \leq \pi$ .

\* Cf. de la Vallée Poussin, loc. cit., p. 370.

† M. Bôcher, *Leçons sur les Méthodes de Sturm*, p. 69.

(d) When  $n$  is sufficiently large so that  $\rho_n^2 > 0$ , the following asymptotic formula holds:\*

$$u_n(x) = \cos \rho_n x + \frac{h \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n} \int_0^x \lambda(t) u_n(t) \sin \rho_n(x-t) dt.$$

(e) If  $\rho_n^2 > 0$ , then†

$$\rho_n = n + \frac{C_n}{n},$$

where  $\rho_n$  denotes the positive square root of  $\rho_n^2$ .

(f) If  $\lambda(x)$  is further restricted so as to possess a continuous derivative in the interval  $0 \leq x \leq \pi$ , the last asymptotic relation is capable of further refinement, as indicated thus:‡

$$\rho_n = n + \frac{C}{n} + \frac{r_n}{n^2},$$

where  $C$  is independent of  $n$ .

(g) Under the hypothesis just stated,§

$$u_n(x) = \cos nx + \frac{\beta(x) \sin nx}{n} + \frac{\alpha(x, n)}{n^2},$$

where  $\beta(x)$  is independent of  $n$ , and has a continuous second derivative in  $(0, \pi)$ .

It will be assumed throughout the remainder of this paper that  $\lambda(x)$  does possess a continuous derivative in the interval  $0 \leq x \leq \pi$ .

2. **Convergence of the Sturm-Liouville interpolating formula.** The proof of the equivalence theorem, which will occupy our attention in the following section, depends in part upon the uniform convergence to the right values of the Sturm-Liouville interpolating development of an analytic function. The demonstration of this fact will be outlined in the present section, although, instead of limiting ourselves to the consideration of analytic functions, we

\* Kneser, *Darstellung willkürlicher Funktionen*, Mathematische Annalen, vol. 58 (1904), pp. 81-147; p. 118.

† Cf. Kneser, loc. cit., p. 120. Throughout this paper, any functional symbol involving  $x$  and one or more integral parameters, either as arguments or subscripts, shall denote a function of  $x$  continuous in the interval  $0 \leq x \leq \pi$  and uniformly bounded for all values of the parameters involved, and, likewise, any letter affected with one or more subscripts shall denote a constant with respect to  $x$  bounded for all values of the subscripts, with the exception of  $\rho_n$ , which is the standard notation for the characteristic numbers.

‡ E. W. Hobson, *On a general convergence theorem, and the theory of the representation of a function by series of normal functions*, Proceedings of the London Mathematical Society, vol. 6 (1908), pp. 349-395; p. 378.

§ Cf. Hobson, loc. cit., p. 378.

shall indicate the proof for a wider class of functions, namely, those satisfying Lipschitz conditions. The method to be used parallels to a large extent that employed by Jackson in establishing the order of convergence of the Sturm-Liouville series.\* The theorem to be proved may be stated as follows:

**THEOREM I.** *If  $f(x)$  satisfies a Lipschitz condition,*

$$|f(x_2) - f(x_1)| \leq \mu |x_2 - x_1|,$$

*throughout the interval  $0 \leq x \leq \pi$ , then*

$$\lim_{p \rightarrow \infty} \sum_p [f(x)] = f(x)$$

*uniformly in the interval.*

The proof consists mainly in obtaining a suitable dominating expression for the difference  $\alpha_{np} u_n(x) - a_{np} \cos nx$ . The essential properties of the coefficients  $\alpha_{np}$  are brought out in the following lemmas.

**LEMMA I.** *If  $f(x)$  satisfies a Lipschitz condition*

$$|f(x_2) - f(x_1)| \leq \mu |x_2 - x_1|$$

*throughout the interval  $0 \leq x \leq \pi$ , then†*

$$\begin{aligned} \frac{1}{p} \left| \sum_{k=0}^p f(x_k) \cos nx_k \right| &= \frac{r_{np}}{n}, \\ \frac{1}{p} \left| \sum_{k=0}^p f(x_k) \sin nx_k \right| &= \frac{r_{np}}{n} \quad (n=1, 2, \dots, p). \end{aligned}$$

The method by which these results are obtained is set forth in a paper by Jackson,‡ although under somewhat different conditions with regard to the function  $f(x)$  and to the length of interval over which the summation is extended. The proof, as adapted to the particular situation under consideration, is similar in character.

\* Jackson, *On the degree of convergence of Sturm-Liouville series*, these Transactions, vol. 15 (1914), pp. 439-466; see pp. 453-456.

† When no confusion is likely, the same letter may be used to denote different constants or functions, subject to the conditions laid down in a previous footnote.

‡ D. Jackson, *On the order of magnitude of the coefficients in trigonometric interpolation*, these Transactions, vol. 21 (1920), pp. 321-332; pp. 323, 324.

LEMMA II. For all values of  $p > n > 0$ ,

$$\left[ \sum_{k=0}^p u_n^2(x_k) \right]^{-1} = \begin{cases} \frac{2}{p} \left( 1 + \frac{r_{np}}{n^2} \right) & (n=1, 2, \dots, s-1), \\ \frac{2}{p} \left( 1 + \frac{r_{np}}{p} \right) & (n=s, s+1, \dots, p-1), \end{cases}$$

where  $s$  denotes  $p/2$  or  $(p+1)/2$ , according as  $p$  is even or odd. When  $n = p$ , the factor  $2/p$  must be replaced by  $1/p$ .

The expression  $\sum' u_n^2(x_k)$  is different from zero in all cases, since  $u_n(0) = 1$ . Squaring both sides of the equality

$$u_n(x) = \cos nx + \frac{\beta(x) \sin nx}{n} + \frac{\alpha(x, n)}{n^2},$$

we obtain an expression for  $u_n^2(x)$  of the form

$$u_n^2(x) = \cos^2 nx + \frac{\beta(x) \sin 2nx}{n} + \frac{\gamma(x, n)}{n^2}.$$

Writing  $\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$  and performing the indicated summation, we obtain, for  $n = 1, 2, \dots, p-1$ ,

$$(3) \quad \sum_{k=0}^p u_n^2(x_k) = p \left( \frac{1}{2} + \frac{1}{np} \sum_{k=0}^p \beta(x_k) \sin 2nx_k + \frac{r_{np}}{n^2} \right),$$

since  $\sum' \cos 2nx_k = 0$ . The modification for the case  $n = p$  is apparent, and need not be explicitly mentioned further. Let us consider the sum

$$\frac{1}{p} \sum_{k=0}^p \beta(x_k) \sin 2nx_k.$$

For  $n = 1, 2, \dots, s-1$ , the fact that  $2n < p$ , together with the additional fact that  $\beta'(x)$  is continuous, permits the application of the preceding lemma to the sum in question, enabling us to write

$$\frac{1}{p} \sum_{k=0}^p \beta(x_k) \sin 2nx_k = \frac{r_{np}}{n} \quad (n=1, 2, \dots, s-1),$$

and, from (3),

$$\sum_{k=0}^p u_n^2(x_k) = p \left( \frac{1}{2} + \frac{r_{np}}{n^2} \right) \quad (n=1, 2, \dots, s-1).$$

For  $n = s, s + 1, \dots, p - 1$ , the factor  $1/n$  appearing in (3) is at most equal to  $2/p$ , consequently

$$\sum_{k=0}^p {}' u_n^2(x_k) = p \left( \frac{1}{2} + \frac{r_{np}}{p} \right) \quad (n = s, s+1, \dots, p-1).$$

We are now prepared to consider the reciprocal of the sum. Choose  $N$  sufficiently large, so that

$$\left| \frac{r_{np}}{n^2} \right| < \frac{1}{4}, \quad \left| \frac{r_{np}}{p} \right| < \frac{1}{4},$$

for all values of  $n$  and  $p$  subject to the inequality  $p > n \geq N$ . It is then legitimate, for these values of the indices, to write the reciprocals in the form\*

$$\left[ \sum_{k=0}^p {}' u_n^2(x_k) \right]^{-1} = \begin{cases} \frac{2}{p} \left( 1 + \frac{r_{np}}{n^2} \right) & (n = N, N+1, \dots, s-1) \\ \frac{2}{p} \left( 1 + \frac{r_{np}}{p} \right) & (n = s, s+1, \dots, p-1). \end{cases}$$

For the remaining values of  $n$ , ranging from 1 to  $N - 1$ , inclusive, we can choose  $r_{np}$ , bounded for all values of  $p > n$ , so that the above expression for  $[\sum {}' u_n^2(x_k)]^{-1}$  still remains valid. The proof of this assertion is essentially contained in the facts that  $u_n(x)$  is continuous, and that  $\sum {}' u_n^2(x_k) \geq 1$ , since  $u_n(x)$  is real and  $u_n(0) = 1$ .

LEMMA III. For  $0 \leq x \leq \pi$ ,

$$|\alpha_{np} u_n(x) - a_{np} \cos nx| < \frac{C}{n^2} \quad (n = 1, 2, \dots, p),$$

where

$$\alpha_{np} = \frac{\sum_{k=0}^p {}' f(x_k) u_n(x_k)}{\sum_{k=0}^p {}' u_n^2(x_k)},$$

$$a_{np} = \frac{2}{p} \sum_{k=0}^p {}' f(x_k) \cos nx_k,$$

$C$  being independent of  $p$  and  $n$ .

\* For present purposes it will suffice to use a less refined form of this equality, namely,

$$\left[ \sum_{k=0}^p {}' u_n^2(x_k) \right]^{-1} = \frac{2}{p} \left( 1 + \frac{r_{np}}{n} \right) \quad (n = 1, 2, \dots, p-1),$$

but the proof of the equivalence theorem, in the next section, demands the more elaborate form.

With the aid of the asymptotic formula for  $u_n(x)$  and the preceding lemma we may write, for  $n = 1, 2, \dots, p-1$ ,

$$\begin{aligned} \alpha_{np} = & 2 \left( 1 + \frac{r_{np}}{n} \right) \left[ \frac{1}{p} \sum_{k=0}^p f(x_k) \cos nx_k \right. \\ & + \frac{1}{np} \sum_{k=0}^p f(x_k) \beta(x_k) \sin nx_k \\ & \left. + \frac{1}{n^2 p} \sum_{k=0}^p f(x_k) \alpha(x_k, n) \right]. \end{aligned}$$

By Lemma I we obtain, in the product of the bracketed terms, a number of quantities of order  $1/n^2$ , the sum of which we denote by  $(r_{np})/n^2$ . Hence

$$\alpha_{np} = \frac{2}{p} \sum_{k=0}^p f(x_k) \cos nx_k + \frac{r_{np}}{n^2}.$$

Multiplying through by  $u_n(x)$ , expressed in its asymptotic form, and again collecting the terms of order  $1/n^2$ , we obtain finally

$$\begin{aligned} \alpha_{np} u_n(x) &= \frac{2}{p} \cos nx \sum_{k=0}^p f(x_k) \cos nx_k + \frac{r_{np}(x)}{n^2} \\ &= a_{np} \cos nx + \frac{r_{np}(x)}{n^2} \quad (n=1, 2, \dots, p-1). \end{aligned}$$

The lemma follows directly from the last equality. The proof for  $n = p$  is obtained by replacing  $2/p$  by  $1/p$  at the appropriate stages in the discussion.

The subsequent procedure consists in expressing  $f(x) - \sum_p[f(x)]$  as the sum of certain differences which can be made arbitrarily small by a proper choice of  $p$  and a subsidiary index  $N$ . These differences will involve, besides terms of  $\sum_p[f(x)]$ , also terms from the sums  $\sigma_p[f(x)]$ ,  $T_p[f(x)]$ , and  $t_p[f(x)]$ , and, in order to simplify the notation, we let

$$\sum_r^s, \quad \sigma_r^s, \quad T_r^s, \quad t_r^s$$

denote the sums of the terms of orders  $r$  to  $s$  inclusive, of the respective formulas. The following inequality will be employed:

$$\begin{aligned} \left| f(x) - \sum_p[f(x)] \right| &\leq \left| \sigma_0^N - \sum_0^N \right| + \left| T_{N+1}^p - \sum_{N+1}^p \right| + \left| T_0^N - t_0^N \right| \\ &\quad + \left| t_0^N - f(x) \right| + \left| f(x) - T_0^p \right| + \left| f(x) - \sigma_0^N \right|, \end{aligned}$$

where  $N$  is an integer presently to be determined. Let the six terms of the right-hand member be denoted by  $D_1, D_2, \dots, D_6$ . Select any  $\epsilon > 0$ ; then choose  $N$  sufficiently large so that

$$D_2 < \frac{\epsilon}{6}, \quad D_4 < \frac{\epsilon}{6}, \quad D_6 < \frac{\epsilon}{6},$$

for all values of  $p \geq N + 1$ . Holding  $N$  fast, choose  $P$  so large that

$$D_1 < \frac{\epsilon}{6}, \quad D_3 < \frac{\epsilon}{6}, \quad D_5 < \frac{\epsilon}{6},$$

for all values of  $p \geq P$ . Adding these inequalities, we arrive at the conclusion that, corresponding to any  $\epsilon > 0$ , there exists an integer  $P$ , such that

$$|f(x) - \sum_p [f(x)]| < \epsilon, \quad p \geq P.$$

It remains to justify these inequalities. Applying Lemma III to  $D_2$ , we find without difficulty that

$$\left| T_p - \sum_{N+1}^p \right| < \frac{C}{N}.$$

The inequalities governing  $D_4, D_6$ , and  $D_6$  depend on the uniform convergence to  $f(x)$  of  $\{f(x)\}^*, T_p[f(x)]^\dagger$  and  $\sigma[f(x)]^\ddagger$  respectively. In regard to  $D_1$  and  $D_3$ , we are dealing essentially with the difference between the integral of a continuous function and the finite sum which tends toward the integral as a limit. Since only a finite number  $N$  of terms are involved, the conclusion is valid.

**3. The equivalence theorem.** Let  $v_0(x), v_1(x), \dots$ , and  $\bar{v}_0(x), \bar{v}_1(x), \dots$ , represent two function systems, each orthogonal in the interval  $a \leq x \leq b$ . The statement that the two series§

$$f(x) \sim \sum_{n=0}^{\infty} b_n v_n(x), \quad f(x) \sim \sum_{n=0}^{\infty} \bar{b}_n \bar{v}_n(x)$$

\* As the theorem on the convergence of Sturm-Liouville series is needed in any event (cf. footnote †), it is perhaps simplest in this connection merely to point out once more that the cosine series is a special case of the Sturm-Liouville series.

† Cf., e.g., Jackson, these Transactions, vol. 14, loc. cit., see pp. 455, 456. The passage cited deals, to be sure, with the case in which an interval of length  $2\pi$  is divided into an odd number of equal parts, but the same method of treatment applies to the problem involved here, which, it will be remembered, is essentially that of representing an even function of period  $2\pi$ , with subdivision of a period interval into an even number of equal parts.

‡ Jackson, these Transactions, vol. 15, loc. cit., see p. 453.

§ The symbol  $\sim$  signifies that the series represents the formal expansion of  $f(x)$ .

possess "essentially the same convergence properties," according to Walsh,\* means that the series

$$\sum_{n=0}^{\infty} [b_n v_n(x) - \bar{b}_n \bar{v}_n(x)]$$

converges absolutely and uniformly to zero throughout the interval  $a \leq x \leq b$ . In his papers† on the subject he adduces two cases where the expansions of  $f(x)$  based on two function systems are equivalent‡; in one case the systems consist respectively of the sine functions

$$\sqrt{2} \sin k\pi x \quad (k=1, 2, \dots),$$

and the normalized solutions of the differential system§

$$\begin{aligned} u''(x) + [\rho^2 - g(x)] u(x) &= 0, \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned}$$

but as yet the equivalence theorem has not been extended to the case where the function systems consist respectively of the cosine functions and the general Sturm-Liouville functions.

If, however, we widen the significance of the term "equivalence" by dropping the restrictions that  $\sum [b_n v_n(x) - \bar{b}_n \bar{v}_n(x)]$  shall converge *absolutely*, we then possess an equivalence theorem, due to Haar,|| for the expansions of  $f(x)$  in terms of the cosine and Sturm-Liouville functions, respectively. It is our purpose here to establish an analogous theorem relative to the expansions of  $f(x)$  by means of the corresponding interpolating formulas. This theorem is quite directly deducible from another more general conclusion, which may properly be introduced as a separate theorem. The statement of the latter is as follows:

**THEOREM II.** *If  $f(x)$  is defined and bounded in the interval  $0 \leq x \leq \pi$ , there exists a constant  $C'$ , independent of  $p$  and  $f(x)$ , such that, for all values of  $p$ ,*

$$|\sum_p [f(x)] - T_p[f(x)]| < C'M,$$

where  $M = \max |f(x)|$  in the interval.

\* J. L. Walsh, *A generalization of the Fourier cosine series*, these Transactions, vol. 22 (1921), pp. 230-239; see p. 236.

† Cf. Walsh, loc. cit.; also, *On the convergence of the Sturm-Liouville series*, Annals of Mathematics, (2), vol. 24 (1922), pp. 109-120; pp. 117-120.

‡ In the sense of "possessing the same convergence properties."

§ This is essentially a limiting case of the differential system (I) for  $h=H=\infty$ .

|| Haar, *Über orthogonale Funktionensysteme*, Mathematische Annalen, vol. 69 (1910), pp. 331-371; p. 335.

We have

$$(4) \quad \sum_p [f(x)] - T_p[f(x)] = \sum_{n=0}^p [\alpha_{np} u_n(x) - a_{np} \cos nx],$$

where, for  $n = 1, 2, \dots, p-1$ ,

$$\begin{aligned} \alpha_{np} u_n(x) - a_{np} \cos nx \\ = \frac{\sum_{k=0}^p f(x_k) u_n(x_k)}{\sum_{k=0}^p u_n^2(x_k)} u_n(x) - \frac{2}{p} \sum_{k=0}^p f(x_k) \cos nx_k \cos nx, \end{aligned}$$

and corresponding expressions hold when  $n = 0$  and  $n = p$ , except that the factor  $2/p$  must be replaced by  $1/p$ . Let us substitute this expression for  $\alpha_{np} u_n(x) - a_{np} \cos nx$  in (4), reverse the order of the resulting double summation with respect to  $n$  and  $k$ , then divide the factor of  $f(x_k)$  through by  $2/p$  and represent the resulting denominator  $(2/p) \sum u_n^2(x_k)$  by the symbol  $S_{np}$ . Thus we obtain

$$\begin{aligned} \sum_p [f(x)] - T_p[f(x)] &= \frac{2}{p} \sum_{k=0}^p f(x_k) \left[ \left\{ \frac{u_0(x_k) u_0(x)}{S_{0p}} - \frac{1}{2} \right\} \right. \\ (5) \quad &+ \sum_{n=1}^{p-1} \left\{ \frac{u_n(x_k) u_n(x)}{S_{np}} - \cos nx_k \cos nx \right\} \\ &\left. + \left\{ \frac{u_p(x_k) u_p(x)}{S_{pp}} - \frac{1}{2} \cos px_k \cos px \right\} \right]. \end{aligned}$$

Denoting the terms in braces by  $v_n(x, k, p)$  and their sum with respect to  $n$  by  $F(x, k, p)$ , we may write (5) in the form

$$\begin{aligned} \sum_p [f(x)] - T_p[f(x)] \\ (6) \quad &= \frac{2}{p} \sum_{k=0}^p f(x_k) \left[ v_0(x, k, p) + \sum_{n=1}^{p-1} v_n(x, k, p) + v_p(x, k, p) \right] \\ &= \frac{2}{p} \sum_{k=0}^p f(x_k) F(x, k, p). \end{aligned}$$

This quantity  $F(x, k, p)$ , which is independent of  $f(x)$ , possesses an important property which leads directly to the theorem. This property is established in the following lemma.

LEMMA IV. *There exists a constant  $Q$ , independent of  $k$  and  $p$ , such that*

$$|F(x, k, p)| < Q,$$

for  $0 \leq x \leq \pi$ , and for all values of  $p$  and of  $k \leq p$ .

From (6), we have

$$F(x, k, p) = v_0(x, k, p) + \sum_{n=1}^{p-1} v_n(x, k, p) + v_p(x, k, p).$$

We need consider only the sum, for the single terms  $v_0(x, k, p)$  and  $v_p(x, k, p)$  are readily seen to be bounded, when it is recalled that  $u_0(0) = 1$  and  $u_0(x)$  is continuous, and (Lemma II) that  $\lim_{p \rightarrow \infty} S_{pp} = 2$ . The product  $u_n(x_k)u_n(x)$ , which is involved in  $v_n(x, k, p)$ , can be expanded by means of the asymptotic formula for  $u_n(x)$  into the form

$$\begin{aligned} \cos nx_k \cos nx + \frac{\beta_1(x, k)}{n} \sin n(x_k + x) \\ + \frac{\beta_2(x, k)}{n} \sin n(x_k - x) + \frac{\delta(x, k, n)}{n^2}, \end{aligned}$$

where

$$\begin{aligned} \beta_1(x, k) &= \frac{1}{2} [\beta(x_k) + \beta(x)], \\ \beta_2(x, k) &= \frac{1}{2} [\beta(x_k) - \beta(x)]. \end{aligned}$$

Recalling the definition of  $v_n(x, k, p)$  given by (6), we apply Lemma II to  $S_{np}$  and utilize the expression just worked out for  $u_n(x_k)u_n(x)$ , thereby obtaining

$$\begin{aligned} v_n(x, k, p) &= \left[ 1 + \left\{ \frac{r_{np}}{n^2} \text{ or } \frac{r_{np}}{p} \right\} \right] u_n(x_k)u_n(x) - \cos nx_k \cos nx \\ &= \frac{\beta_1(x, k)}{n} \sin n(x_k + x) + \frac{\beta_2(x, k)}{n} \sin n(x_k - x) \\ &\quad + \left\{ \frac{r_{np}(x, k)}{n^2} \text{ or } \frac{r_{np}(x, k)}{p} \right\}, \end{aligned}$$

the first or second terms in the braces being used according as  $n < s$  or  $n \geq s$ . For the sum we may therefore write

$$\begin{aligned} \sum_{n=1}^{p-1} v_n(x, k, p) &= \beta_1(x, k) \sum_{n=1}^{p-1} \frac{\sin n(x_k + x)}{n} + \beta_2(x, k) \sum_{n=1}^{p-1} \frac{\sin n(x_k - x)}{n} \\ &\quad + \sum_{n=1}^{s-1} \frac{r_{np}(x, k)}{n^2} + \frac{1}{p} \sum_{n=s}^{p-1} r_{np}(x, k). \end{aligned}$$

Each of the sine sums, expressed in terms of a variable  $y = x_k \pm x$ , is the partial sum of a well known convergent Fourier expansion, and is bounded\* for all values of  $p$  and of the arguments  $x_k \pm x$ . The remaining sums are obviously bounded likewise. Hence  $F(x, k, p)$  must be dominated in absolute value by some constant  $Q$ , for all values of  $p$ , and all values of  $x$  and  $x_k$  in the interval.

The theorem follows directly, for we may write

$$\left| \sum_p [f(x)] - T_p[f(x)] \right| \leq \frac{2}{p} \sum_{k=0}^p \left| f(x_k) F(x, k, p) \right| < 2MQ = C'M.$$

This theorem shows, then, that as far as mere boundedness is concerned,  $T_p[f(x)]$  and  $\sum_p [f(x)]$  behave in a similar manner, provided only that  $f(x)$  is defined and bounded. If the additional restriction of continuity is imposed upon  $f(x)$ , we obtain very easily our equivalence theorem, which may be stated thus:

**THEOREM III.** *If  $f(x)$  is continuous,  $0 \leq x \leq \pi$ , then*

$$\lim_{p \rightarrow \infty} \left\{ \sum_p [f(x)] - T_p[f(x)] \right\} = 0$$

*uniformly in the interval.*

Let  $f_1(x), f_2(x), \dots$  represent a sequence of analytic functions, such that

$$\lim_{p \rightarrow \infty} [f(x) - f_p(x)] = 0$$

uniformly for  $0 \leq x \leq \pi$ . Let  $\delta_p(x) = f(x) - f_p(x)$ . Then

$$\begin{aligned} (7) \quad & \left| \sum_p [f(x)] - T_p[f(x)] \right| \\ & \leq \left| \sum_p [f_p(x)] - T_p[f_p(x)] \right| \\ & \quad + \left| \sum_p [\delta_p(x)] - T_p[\delta_p(x)] \right|. \end{aligned}$$

Choose  $\epsilon > 0$ . Then there exists a value of  $p$ , say  $N$ , such that

$$|\delta_N(x)| \leq \frac{\epsilon}{2C'}.$$

By the preceding theorem, therefore,

$$\left| \sum_p [\delta_N(x)] - T_p[\delta_N(x)] \right| < C' \frac{\epsilon}{2C'} = \frac{\epsilon}{2}.$$

\* Cf., e. g., D. Jackson, *Über eine trigonometrische Summe*, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 257-262; see p. 257.

With  $N$  fixed, we can choose an integer  $P$ , such that

$$|\sum_p [f_N(x)] - T_p[f_N(x)]| < \frac{\epsilon}{2},$$

for all values of  $p \geq P$ . This choice of  $P$  is rendered possible because  $f_N(x)$ , being analytic, is capable of representation by both the trigonometric and Sturm-Liouville interpolating formulas with errors arbitrarily small. With the application of the last two inequalities to (7), the theorem follows directly:

$$|\sum_p [f(x)] - T_p[f(x)]| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad p \geq P.$$

An immediate corollary is that if  $f(x)$  is any continuous function for which  $T_p[f(x)]$  converges uniformly to  $f(x)$ , then  $\sum_p [f(x)]$  will do the same. In particular, a sufficient condition for convergence is that  $f(x)$  satisfy the familiar Lipschitz-Dini condition.\*

It may be added, in passing, that we have in this section the materials with which to demonstrate the existence of a Sturm-Liouville interpolating formula which converges for all continuous functions. It is formed by analogy with the Fejér mean,† but is not identical with the arithmetical mean of the sums  $\sum_0[f(x)]$ ,  $\sum_1[f(x)]$ ,  $\dots$ ,  $\sum_p[f(x)]$ . We shall merely state the facts, without proof, in the following theorem:

**THEOREM IV.** *If  $f(x)$  is continuous,  $0 \leq x \leq \pi$ , then the interpolating formula*

$$\bar{\sum}_p [f(x)] = \bar{\alpha}_{0,p} u_0(x) + \bar{\alpha}_{1,p} u_1(x) + \dots + \bar{\alpha}_{p-1,p} u_{p-1}(x),$$

where

$$\bar{\alpha}_{n,p} = \frac{p-n}{p} \alpha_{n,p} \quad (n=0, 1, \dots, p-1),$$

converges uniformly to  $f(x)$  throughout the interval  $0 \leq x \leq \pi$ .

#### 4. Degree of convergence of the Sturm-Liouville interpolating formula.

In the course of the discussion concerning the degree of convergence of the

\* Cf. Faber, loc. cit., p. 422; D. Jackson, these Transactions, vol. 14, loc. cit., p. 456.

† Cf. D. Jackson, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371-375; p. 372.

trigonometric interpolating formula,\* we find reference made to the fact that the trigonometric interpolating expansion of a finite trigonometric sum is identically that sum, a consequence of the well known fact that the trigonometric functions are, if we may use the term in this connection, orthogonal with respect to summation over the interval  $(0, 2\pi)$ . This fact, in the case of the cosine formula, may be expressed in the form

$$\sum_{k=0}^p \cos mx_k \cos nx_k = 0, \quad m \neq n \quad (m \leq p, n \leq p).$$

In the case of the Sturm-Liouville functions, however,  $\sum u_m(x_k)u_n(x_k)$  is not generally equal to zero, but only tends toward zero as  $p$  increases, hence the Sturm-Liouville interpolating formula for a finite sum is not identically that sum, but only an approximation to it. The degree of this approximation can, however, be determined; the conditions and solution of the problem thus suggested find precise formulation in the following

**THEOREM V.** *Given an infinite sequence of functions  $\varphi_p(x)$  of the type*

$$\varphi_p(x) = c_{0p}u_0(x) + c_{1p}u_1(x) + \cdots + c_{pp}u_p(x) \quad (p=1, 2, \cdots),$$

*and a constant  $K$ , independent of  $p$ , such that*

$$|\varphi_p(x)| < K, \quad 0 \leq x \leq \pi \quad (p=1, 2, \cdots),$$

*and another constant  $A$ , independent of  $n$  and  $p$ , such that*

$$|c_{0p}| < A, \quad |c_{np}| < \frac{A}{n}, \quad 1 \leq n \leq p \quad (p=1, 2, \cdots);$$

*then there exists a constant  $C$ , independent of  $p$ , such that*

$$|\sum_p [\varphi_p(x)] - \varphi_p(x)| < \frac{C}{p}, \quad 0 \leq x \leq \pi \quad (p=1, 2, \cdots).$$

The complete proof of this theorem is quite involved and tedious; it seems best, therefore, to present in this paper the mere outline of the proof, containing some of the more important subordinate results, and other details sufficient to indicate the methods employed.

If we introduce the notation

$$\sum_{n=0}^p m y_n = \left[ \sum_{n=0}^p y_n \right] - y_m$$

\* D. Jackson, these Transactions, vol. 14, loc. cit., p. 455.

we can express the difference  $\sum_p [\varphi_p(x)] - \varphi_p(x)$  in the compact form

$$\sum_p [\varphi_p(x)] - \varphi_p(x) = \frac{\sum_{m=0}^p u_m(x) \sum_{n=0}^p {}^m c_{np} \sum_{k=0}^p {}' u_m(x_k) u_n(x_k)}{\sum_{k=0}^p {}' u_m^2(x_k)}.$$

It will be found convenient to introduce the additional notation

$$S(m, n, p) = \frac{\pi}{p} \sum_{k=0}^p {}' u_m(x_k) u_n(x_k),$$

and to separate out the terms with indices  $m = 0$  and  $n = 0$ , since these terms cannot be represented by the asymptotic formula. We then have

$$\begin{aligned} (8) \quad \sum_p [\varphi_p(x)] - \varphi_p(x) &= u_0(x) \frac{\sum_{n=1}^p c_{np} S(0, n, p)}{S(0, 0, p)} \\ &+ \sum_{m=1}^p u_m(x) \frac{c_{0p} S(m, 0, p)}{S(m, m, p)} + \sum_{m=1}^p u_m(x) \frac{\sum_{n=1}^p c_{np} S(m, n, p)}{S(m, m, p)}. \end{aligned}$$

The problem presented is essentially that of determining the order of magnitude of the quantities  $S(m, n, p)$ .

To do this, we break up  $S(m, n, p)$  into parts corresponding to the several terms of the product  $u_m(x)u_n(x)$ , when the characteristic functions have been replaced by their asymptotic representations. The coefficients of the sine terms in the product, which appear below, will be separated into linear functions and functions vanishing at the end points 0 and  $\pi$ :

$$\frac{1}{2} \beta(x) = (a + bx) + \eta(x),$$

$$\beta(x) \alpha(x, n) = (a_n + b_n x) + \eta_n(x),$$

where  $\eta(0) = \eta_n(0) = \eta(\pi) = \eta_n(\pi) = 0$ . Making these substitutions, and effecting certain trigonometric reductions, we obtain for  $u_m(x)u_n(x)$  the following expression of eleven terms:

$$\begin{aligned}
u_m(x) u_n(x) &= \frac{1}{2} [\cos(m+n)x + \cos(m-n)x] \\
&+ (a+bx) \left[ \frac{\sin(m+n)x + \sin(m-n)x}{m} + \frac{\sin(m+n)x - \sin(m-n)x}{n} \right] \\
&+ \eta(x) \left[ \frac{\sin(m+n)x + \sin(m-n)x}{m} + \frac{\sin(m+n)x - \sin(m-n)x}{n} \right] \\
&+ \frac{\beta^2(x)}{2mn} [\cos(m-n)x - \cos(m+n)x] + \frac{\alpha(x, n)}{n^2} \cos mx + \frac{\alpha(x, m)}{m^2} \cos nx \\
&+ \frac{(a_n + b_n x) \sin mx}{n^2 m} + \frac{(a_m + b_m x) \sin nx}{m^2 n} \\
&+ \frac{\eta_n(x) \sin mx}{n^2 m} + \frac{\eta_m(x) \sin nx}{m^2 n} + \frac{\alpha(x, n) \alpha(x, m)}{n^2 m^2}.
\end{aligned}$$

Let these eleven terms be denoted by  $g_r(x, m, n)$ ,  $r = 1, 2, \dots, 11$ , respectively, so that

$$u_m(x) u_n(x) = \sum_{r=1}^{11} g_r(x, m, n).$$

Since

$$\int_0^\pi u_m(x) u_n(x) dx = 0,$$

it is clear that

$$\begin{aligned}
\frac{\pi}{p} \sum_{k=0}^p u_m(x_k) u_n(x_k) &= \frac{\pi}{p} \sum_{k=0}^p u_m(x_k) u_n(x_k) - \int_0^\pi u_m(x) u_n(x) dx \\
&= \sum_{r=1}^{11} \left[ \frac{\pi}{p} \sum_{k=0}^p g_r(x_k, m, n) - \int_0^\pi g_r(x, m, n) dx \right].
\end{aligned}$$

Denoting the bracketed quantity by  $S_r(m, n, p)$ , and recalling the notation used for the left-hand member, we can write

$$S(m, n, p) = \sum_{r=1}^{11} S_r(m, n, p).$$

The determination of the orders of magnitude of the quantities  $S_r(m, n, p)$  involves the use of a number of more or less well known formulas and theorems, which may be summarized thus:

$$(a) \quad \sum_{k=0}^p \cos \nu x_k = \begin{cases} 0, & \nu \neq 2lp, \\ p, & \nu = 2lp, l=0, 1, 2, \dots; \end{cases}$$

$$(b) \quad \sum_{k=0}^p \sin \nu x_k = \begin{cases} 0, & \nu \text{ even}, \\ \cot \frac{\nu\pi}{2p}, & \nu \text{ odd}; \end{cases}$$

$$(c) \quad \sum_{k=0}^p x_k \sin \nu x_k = \begin{cases} 0, & \nu = 2lp, \\ \frac{-\pi(-1)^l}{2} \cot \frac{\nu\pi}{2p}, & \nu \neq 2lp; \end{cases}$$

$$(d) \quad \int_0^\pi \cos \nu x \, dx = \begin{cases} 0, & \nu \neq 0, \\ \pi, & \nu = 0; \end{cases}$$

$$(e) \quad \int_0^\pi \sin \nu x \, dx = \begin{cases} 0, & \nu \text{ even}, \\ \frac{2}{\nu}, & \nu \text{ odd}; \end{cases}$$

$$(f) \quad \int_0^\pi x \sin \nu x \, dx = \begin{cases} 0, & \nu = 0, \\ -(-1)^l \frac{\pi}{\nu}, & \nu \neq 0; \end{cases}$$

(g) if  $f''(x)$  is continuous in the interval  $0 \leq x \leq \pi$ , then  $f(x)$  can be expanded in a Fourier series of cosines with coefficients of order\*  $1/\nu^2$ ;

(h) if  $f(x)$  satisfies the above hypothesis and the additional condition that  $f(0) = f(\pi) = 0$ , then it can be expanded into a Fourier sine series with coefficients of order  $1/\nu^2$ .

For  $S_1(m, n, p)$  and  $S_2(m, n, p)$ , we obtain explicit expressions. From (a) and (d) it follows that  $S_1(m, n, p) = 0$ .

From (b), (c), (e), and (f), we obtain

$$S_2(m, n, p) = \frac{\pi}{p} H_{mn} \frac{(1/m) \sin(m\pi/p) - (1/n) \sin(n\pi/p)}{\sin[(m+n)\pi/(2p)] \sin[(m-n)\pi/(2p)]},$$

where  $H_{mn} = -b\pi/2$  or  $a + (b\pi/2)$  according as  $m+n$  is even or odd.

\* Cf. Picard, *Traité d'Analyse*, vol. 1, pp. 255, 256, in edition 2, or p. 334 in edition 3.

For each of the remaining quantities  $S_r(m, n, p)$ , we obtain a dominating expression which indicates its order of magnitude. The manner in which this is obtained may be briefly outlined. Leaving  $g_7(x, m, n)$ ,  $g_8(x, m, n)$ , and  $g_{11}(x, m, n)$  out of consideration for the present, we notice that each term  $g_r(x, m, n)$  involves either a sine or cosine term as one of its factors. We proceed to expand the other factor into a Fourier series; in the latter case, into a cosine series, and in the former, into a sine series; and then we change the resulting products into the sums and differences of cosines. This, of course, is equivalent to expanding the function  $g_r(x, m, n)$  itself into a cosine series. In the case of  $g_{11}(x, m, n)$ , we expand the product  $\alpha(x, n)\alpha(x, m)$  into a cosine series directly. Recalling that

$$S_r(m, n, p) = \frac{\pi}{p} \sum_{k=0}^p g_r(x_k, m, n) - \int_0^{\pi} g_r(x, m, n) dx,$$

we apply (a) and (d) to the sum and integral, respectively, whereupon all except one out of every  $2p$  terms in the expansion of  $g_r(x, m, n)$  disappear, leaving  $S_r(m, n, p)$  in the form of an infinite series whose sum approaches zero as  $p$  increases indefinitely. By the theorems enunciated in (g) and (h), we know the orders of magnitude of the terms of this series, hence we can determine that of  $S_r(m, n, p)$ .

It should be mentioned here that the function  $\alpha(x, n)/n^2$ , which is involved indirectly in  $S_9$  and  $S_{10}$ , and directly in  $S_5$ ,  $S_6$  and  $S_{11}$ , possesses a continuous second derivative, uniformly bounded for all values of  $n$ . In dealing with  $S_5$  and  $S_6$ , we must know in some detail how the derivatives of  $\alpha(x, n)/n^2$  depend upon  $n$ . The nature of this dependence is indicated by the relations

$$\begin{aligned} \frac{d}{dx} \left[ \frac{\alpha(x, n)}{n^2} \right] &= \frac{P(x, n)}{n}, \\ \frac{d^2}{dx^2} \left[ \frac{\alpha(x, n)}{n^2} \right] &= Q_1(x, n) \cos nx + Q_2(x, n) \sin nx + \frac{Q_3(x, n)}{n}, \end{aligned}$$

where  $Q_1'(x, n)$  and  $Q_2'(x, n)$ , as well as  $Q_1$ ,  $Q_2$ , and  $Q_3$ , are uniformly bounded and continuous.

The remaining terms,  $g_7(x, m, n)$  and  $g_8(x, m, n)$ , we treat the same as  $g_2$  obtaining thereby explicit expressions for  $S_7(m, n, p)$  and  $S_8(m, n, p)$ , for which suitable dominating quantities are easily found.

The results obtained through the processes thus briefly sketched appear in the following inequalities:\*

\* Since  $m \neq n$ ,  $m+n$  can never equal  $2p$ .

$$|S_3(m, n, p)| < \frac{4LV_1}{p^2} \left[ \frac{1}{m} + \frac{1}{n} \right] + \frac{V_1}{(2p - m - n)^2} \left[ \frac{1}{m} + \frac{1}{n} \right],$$

$$|S_4(m, n, p)| < \frac{1}{mn} \left[ \frac{2LV_2}{p^2} + \frac{V_2}{2(2p - m - n)^2} \right],$$

$$|S_5(m, n, p)| < \frac{4LV_3}{np^2} + \frac{V_3}{p^2(2p - m - n)},$$

$$|S_7(m, n, p)| < \frac{V_4}{n^2 p^2},$$

$$|S_9(m, n, p)| < \frac{2LV_5}{mp^2},$$

$$|S_{11}(m, n, p)| < \frac{LV_6}{p^2} \left[ \frac{1}{m} + \frac{1}{n} \right].$$

There is no need for writing down separate inequalities for  $S_6$ ,  $S_8$ , and  $S_{10}$ , since they are, in form, entirely analogous to those for  $S_5$ ,  $S_7$ , and  $S_9$ , respectively. The letter  $L$  denotes  $\sum (1/n^2)$ ,  $n = 1, 2, \dots$ , and the  $V$ 's denote constants whose values are related to the maximum values of the derivatives of the functions which were expanded in the Fourier cosine or sine series.

Referring back to (8) and recalling that  $u_n(x)$  remains bounded for all values of  $n$ , that  $S(m, n, p)$  has a positive lower bound independent of  $m$  and  $p$ , and that the coefficients  $c_{np}$  are of order  $1/n$ , we find that our next problem is essentially that of determining suitable upper bounds for the double summations

$$\sum_{n=1}^p \sum_{m=1}^p \frac{1}{n} |S_r(m, n, p)| \quad (r=3, 4, \dots, 11).$$

We notice that the expressions which dominate  $|S_r(m, n, p)|$  are simple functions of the discrete variables  $m$  and  $n$ , so that the double sums can be replaced by double integrals which are easily evaluated, yielding thereby the desired upper bounds. By this method we are enabled to deduce the existence of a constant, say  $W$ , such that, for all values of  $p \geq 1$ ,

$$(9) \quad \sum_{m=1}^p \sum_{n=1}^p \frac{1}{n} |S_r(m, n, p)| < \frac{W}{p} \quad (r=3, 4, \dots, 11).$$

Thus far we have made use of only one of the properties of the coefficients  $c_{rp}$ , namely, that  $|c_{np}| < A/n$ , but, in dealing with  $S_2(m, n, p)$ , we must utilize the other property, namely, that they are such that the sequence of

Sturm-Liouville sums  $\varphi_p(x)$  remains bounded for all values of  $p$ . Hence we must work with the sum

$$\sum_{m=1}^p \left| \sum_{n=1}^p c_{np} S_2(m, n, p) \right|.$$

Recalling the explicit expression previously obtained for  $S_2(m, n, p)$ , we shall find it expedient, by means of a change of variable, to reduce  $S_2(m, n, p)$  to the form of a function of two discrete variables ranging between the limits 0 and  $\pi$ . Since  $x_k = k\pi/p$ , we can write

$$S_2(m, n, p) = \frac{\pi^2}{p^2} H_{mn} \frac{\frac{\sin x_m}{x_m} - \frac{\sin x_n}{x_n}}{\sin \frac{x_m + x_n}{2} \sin \frac{x_m - x_n}{2}},$$

where  $0 < x_m \leq \pi$ ,  $0 < x_n \leq \pi$ ,  $x_m \neq x_n$ . Let  $L(x_m, x_n)$  denote the complex fraction, so that

$$(10) \quad S_2(m, n, p) = \frac{\pi^2}{p^2} H_{mn} L(x_m, x_n).$$

Denoting by  $L(y, z)$  the corresponding function of two continuous variables  $y, z$ , and expanding the numerator of  $L(y, z)$  into a power series, we find that we can write

$$L(y, z) = \frac{\frac{y+z}{2}}{\sin \frac{y+z}{2}} \cdot \frac{\frac{y-z}{2}}{\sin \frac{y-z}{2}} \cdot \xi(y, z),$$

where  $\xi(y, z)$  is analytic in the region  $0 \leq y \leq \pi$ ,  $0 \leq z \leq \pi$ . The properties of  $L(y, z)$  are therefore essentially those of the reciprocals of functions of the familiar type  $(\sin x)/x$ . Since these properties pertain to certain regions of the  $y, z$  domain, denoted by  $R$ , we divide the latter into two subregions  $R_1$  and  $R_2$ , where  $R_2$  is defined by the inequalities  $s \leq m \leq p$ ,  $s \leq n \leq p$ , and  $R_1 = R - R_2$ .

In  $R_1$  the following inequalities hold:

$$(11) \quad |L(y, z)| < L_1,$$

$$(12) \quad \left| \frac{\partial}{\partial z} L(y, z) \right| < G_1,$$

where  $L_1$  and  $G_1$  denote constants, independent of  $m, n$ , and  $p$ . Let

$$\sum_n^m c_{np} S_2(m, n, p)$$

denote the sum, with respect to  $n$ , of  $c_{np} S_2(m, n, p)$  taken over the values of  $n$  for which  $(x_m, x_n)$  belongs to  $R_1$ . By using the expression previously obtained for  $S_2(m, n, p)$  and an obvious identity, we may write

$$\begin{aligned} \frac{p^2}{\pi^2} \sum_n^m c_{np} S_2(m, n, p) &= \sum_n^m c_{np} H_{mn} L(x_m, x_n) \\ (13) \qquad \qquad \qquad &= L(x_m, 0) \sum_n^m c_{np} H_{mn} \\ &\quad + \sum_n^m c_{np} H_{mn} [L(x_m, x_n) - L(x_m, 0)]. \end{aligned}$$

By the law of the mean,

$$L(x_m, x_n) - L(x_m, 0) = \frac{n\pi}{p} \frac{\partial}{\partial z} L(x_m, h_{mn} x_n),$$

where  $0 < h_{mn} < 1$ , and, by (12),

$$|L(x_m, x_n) - L(x_m, 0)| < \frac{n\pi G_1}{p}.$$

Since  $|c_{np}| < A/n$ , we finally obtain

$$(14) \quad \left| \sum_n^m c_{np} H_{mn} [L(x_m, x_n) - L(x_m, 0)] \right| < H \sum_n^m \frac{A}{n} \frac{n\pi G_1}{p} = G_2,$$

where  $H > |H_{mn}|$ . Upon replacing, in the first sum on the right-hand side of (13), the quantities  $H_{mn}$  by their values

$$H_{mn} = \begin{cases} -\frac{b\pi}{2}, & m+n \text{ even,} \\ a + \frac{b\pi}{2}, & m+n \text{ odd,} \end{cases}$$

and condensing the resultant expression, we obtain

$$\begin{aligned} (15) \quad \sum_n^m c_{np} H_{mn} &= (-1)^{m+1} \frac{1}{2} (b\pi + a) \sum_n^m (-1)^n c_{np} \\ &\quad + \frac{1}{2} a \sum_n^m c_{np}. \end{aligned}$$

Since  $\varphi_p(x) = \sum c_{np} u_n(x)$  is uniformly bounded for all values of  $p$  and for all values of  $x$  in the interval  $0 \leq x \leq \pi$ , it is clear that the sequences of constants  $|\varphi_p(0)|$  and  $|\varphi_p(\pi)|$  are bounded. But, from the asymptotic formula for  $u_n(x)$ , we find that

$$u_n(0) = 1,$$

$$u_n(\pi) = (-1)^n + \frac{r_n}{n}, \quad n > 0.$$

Therefore

$$\begin{aligned} \sum_{n=0}^p c_{np} &= \sum_{n=0}^p c_{np} u_n(0) = \varphi_p(0), \\ \sum_{n=0}^p (-1)^n c_{np} &= \sum_{n=0}^p c_{np} u_n(\pi) - c_{0p} r_0 - \sum_{n=1}^p c_{np} \frac{r_n}{n} \\ &= \varphi_p(\pi) - c_{0p} r_0 - \sum_{n=1}^p \frac{r_{np}}{n^2}, \end{aligned}$$

whence it is apparent that  $\sum c_{np}$  and  $\sum c_{np} (-1)^n$  remain respectively bounded for all values of  $p$ . It is easily shown that the same is true of the partial sums

$$\sum_n^m c_{np}, \quad \sum_n^m (-1)^n c_{np},$$

which appear in (15). Hence, combining (13), (14), and (15), we finally deduce the existence of a constant  $W_1$ , such that

$$\left| \sum_n^m c_{np} S_2(m, n, p) \right| < \frac{W_1}{p^2} \quad (p = 1, 2, \dots).$$

Summing this with respect to  $m$  over the range  $1 \leq m \leq p$ , we obtain the desired result, namely

$$(16) \quad \sum_m \left| \sum_n^m c_{np} S_2(m, n, p) \right| < \frac{W_1}{p}.$$

Our next problem is to obtain a similar inequality for this summation extended over the region  $R_2$ , which, as we may recall, constitutes that part of the  $(x_m, x_n)$  domain in which  $m$  and  $n$  are subject to the inequalities  $s \leq m \leq p$ ,  $s \leq n \leq p$ ,  $m \neq n$ . In this region we find that

$$|L(x_m, x_n)| < \frac{\text{const.}}{2\pi - x_m - x_n}.$$

Replacing  $x_m$  and  $x_n$  by their respective values,  $m\pi/p$  and  $n\pi/p$ , and recalling (10), we can write\*

$$\left| S_2(m, n, p) \right| < \frac{K}{p(2p - m - n)},$$

where  $K$  is a constant, independent of  $p$ ,  $m$ , and  $n$ . In working with the double sum

$$\sum_m \left| \sum_n c_{np} S_2(m, n, p) \right|,$$

we shall have no further occasion to utilize that property of the coefficients  $c_{np}$  whereby  $\sum c_{np} u_n(x)$  remains bounded for all values of  $p$ ; all we require is the fact that  $c_{np} < A/n$ . Hence it appears that our problem is essentially that of determining the order of magnitude of

$$\frac{1}{p} \sum_{m=0}^p \sum_{n=0}^p \frac{1}{n(2p - m - n)}.$$

This problem can be further simplified by noting that  $1/n \leq 2/p$ , whereby we may write

$$\sum_m \left| \sum_n c_{np} S_2(m, n, p) \right| < \frac{2AK}{p^2} \sum_{m=0}^p \sum_{n=0}^p \frac{1}{2p - m - n}.$$

Replacing the double sum by the corresponding double integral and evaluating the latter, we arrive at the result

$$\sum_{m=0}^p \sum_{n=0}^p \frac{1}{2p - m - n} < pK'.$$

Combining the last two inequalities, we obtain

$$\sum_m \left| \sum_n c_{np} S_2(m, n, p) \right| < \frac{W_2}{p},$$

which may be combined with (16) to yield the desired conclusion, namely

$$(17) \quad \sum_{m=1}^p \left| \sum_{n=1}^p c_{np} S_2(m, n, p) \right| < \frac{W_3}{p}.$$

We are now in a position to determine the order of magnitude of the quantity appearing on the right-hand side of (8), by which  $|\sum_p [\varphi_p(x)] - \varphi_p(x)|$  is dominated. The inequalities (17) and (9) show that the double

\* Since  $m \neq n$ ,  $m+n$  never reaches  $2p$ .

summation in (8) never exceeds some fixed multiple of  $1/p$ . The same is seen to be true of the single summations in the right-hand member of (8), as a result of reasoning analogous in principle to that which precedes, but materially simpler in execution. Hence the difference  $\sum_p [\varphi_p(x)] - \varphi_p(x)$  must, as the theorem states, be dominated in absolute value by some constant multiple of  $1/p$ .

From this theorem, considered in conjunction with Theorem II, one may derive a conclusion as to the degree of convergence of the Sturm-Liouville formula for a function satisfying a Lipschitz condition, essentially similar in proof to the corresponding theorem in trigonometric interpolation. This conclusion may be stated thus:

**THEOREM VI.** *If  $f(x)$  satisfies a Lipschitz condition,*

$$|f(x_2) - f(x_1)| < \mu(x_2 - x_1), \quad 0 \leq x_1 < x_2 \leq \pi,$$

*then there exists a constant  $G$ , independent of  $p$ , such that*

$$|\sum_p [f(x)] - f(x)| < \frac{G \log p}{p}, \quad p \geq 2,$$

*for all values of  $x$  in the interval  $0 \leq x \leq \pi$ ; and, for the points of interpolation  $x_k, k = 0, 1, \dots, p$ ,*

$$|\sum_p [f(x_k)] - f(x_k)| < \frac{G}{p}, \quad p \geq 1.$$

We shall not carry out the demonstration of this theorem, but merely point out that, by analogy with the trigonometric case, the proof requires the existence of a Sturm-Liouville formula  $\varphi_p(x)$  which shall represent  $f(x)$  with an error less in absolute value than a fixed multiple of  $1/p$ , and whose coefficients  $c_{np}$  shall satisfy the hypotheses of the preceding theorem. It is known, however, that such a formula does exist,\* hence the proof offers no further difficulty.

\* D. Jackson, these Transactions, vol. 15, loc. cit., p. 466.

## SINGULAR RULED SURFACES IN SPACE OF FIVE DIMENSIONS\*

BY

E. B. STOFFER

In a previous paper† the author has made a study of the projective differential properties of ruled surfaces in space of five dimensions by means of a system of two linear homogeneous differential equations of the third order. The method gave results only for *regular*, or *non-singular*, ruled surfaces, that is, for surfaces which do not have in general three consecutive generators in the same 4-space. However, the singular surfaces are of even greater interest than the regular surfaces. This is particularly true because surfaces in 4-space and even in 3-space may be regarded as such singular surfaces and, consequently, generalizations of some of the properties of ruled surfaces in 3-space become quite apparent. In this paper a plan has been devised for studying all ruled surfaces in 5-space except those which have in general consecutive generators intersecting, that is, which are developable in the ordinary sense. Since the regular surfaces in 5-space have already been treated in the paper mentioned, the study in this paper will be confined largely to non-developable singular ruled surfaces, including surfaces in 4-space but not in 3-space.

The method here used may be immediately generalized to provide for the study of any spread generated by  $\infty^1$  linear spaces in space of any number of dimensions, provided merely that two consecutive generating spaces do not in general have a point in common. Moreover, the author in previous papers‡ has set up much of the necessary analytical machinery for such studies.

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† *Invariants of linear homogeneous differential equations, with applications to ruled surfaces in five-dimensional space*, Proceedings of the London Mathematical Society, (2), vol. 11 (1913), pp. 185-224.

‡ *On semivariants of linear homogeneous differential equations*, Proceedings of the London Mathematical Society, (2), vol. 15 (1916), pp. 217-226; *On invariants and covariants of linear homogeneous differential equations*, Proceedings of the London Mathematical Society, (2), vol. 17 (1919), pp. 337-352. This second paper will be referred to in the future as *Invariants and covariants*.

The papers most closely related to the present work are three by Ranum\* published from 1912 to 1915, and one by Bompiani† published in 1914.

### 1. THE EQUATIONS AND THEIR TRANSFORMATIONS

The system of linear homogeneous differential equations

$$(1) \quad y_i'' + 2 \sum_{k=1}^3 p_{ik} y_k' + \sum_{k=1}^3 q_{ik} y_k = 0 \quad (i = 1, 2, 3),$$

where  $p_{ik}$  and  $q_{ik}$  are functions of the independent variable  $x$ , is left invariant in form by the transformations‡

$$(2) \quad x = \xi(x),$$

$$(3) \quad y_i = \sum_{k=1}^3 \alpha_{ik} \bar{y}_k \quad (i = 1, 2, 3),$$

where  $\xi$  and  $\alpha_{ik}$  are arbitrary functions of  $x$ , and where the determinant  $|\alpha_{ik}|$  of the transformation (3) is different from zero. Moreover, these are the most general transformations which leave (1) invariant in form.

Let  $(y_{1i}, y_{2i}, y_{3i})$  ( $i=1, 2, \dots, 6$ ) be a fundamental system of solutions of equations (1). If the six functions  $y_{1i}$  ( $i=1, 2, \dots, 6$ ) of  $x$  are interpreted as the homogeneous coördinates of a point§  $y_1$  in 5-space, this point generates a curve  $C_1$  as  $x$  varies. Likewise, the points  $y_2$  and  $y_3$  generate two curves  $C_2$  and  $C_3$ , respectively. Corresponding points on the three curves, that is, points for which  $x$  has the same value, determine a plane and as  $x$  varies this plane generates a three spread  $S_3$  of which  $C_1, C_2, C_3$  are directrix curves. Since  $(y_{1i}, y_{2i}, y_{3i})$  form a fundamental system of solutions of (1) it is impossible to find six functions  $\lambda_j, \mu_j$  ( $j=1, 2, 3$ ) such that the equations

$$\sum_{j=1}^3 \lambda_j y_{ji} + \sum_{j=1}^3 \mu_j y_{ji}' = 0 \quad (i = 1, 2, \dots, 6)$$

\* A. Ranum, *On the projective differential geometry of  $n$ -dimensional spreads generated by  $\infty^1$  flats*, Annali di Matematica Pura ed Applicata, (3), vol. 19 (1912), pp. 205-249; *On the projective differential classification of  $n$ -dimensional spreads generated by  $\infty^1$  flats*, American Journal of Mathematics, vol. 37 (1915), pp. 117-158; *On the differential geometry of ruled surfaces in 4-space and cyclic surfaces in 3-space*, these Transactions, vol. 16 (1915), pp. 89-110.

† E. Bompiani, *Alcune proprietà proiettivo-differenziali dei sistemi di rette negli iperspazi*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 305-331.

‡ Cf. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906.

§ Where no confusion can arise a point whose coördinates are of the form  $\alpha_i$  ( $i=1, 2, \dots, 6$ ) will be denoted simply by  $\alpha$ .

are verified. This is equivalent to saying that  $S_3$  is regular, that is, that consecutive generating planes of  $S_3$  do not in general have a point in common.

The fact that all fundamental systems of solutions of (1) are linearly related shows that the three spreads obtained from different fundamental systems are all projectively related.

If we now consider the line determined by corresponding points of two of the curves, say  $C_1$  and  $C_2$ , there is generated, as  $x$  varies, a ruled surface  $S$  which is on  $S_3$  and of which  $C_1$  and  $C_2$  are directrix curves. Since consecutive generating planes of  $S_3$  cannot have a point in common, consecutive generators of  $S$  cannot have a point in common. In other words,  $S$  cannot be developable in the ordinary sense.

Any non-developable ruled surface in 5-space may be considered as being on a regular three spread. Any two directrix curves of the ruled surface may be selected as directrix curves of a three spread  $S_3$  and any other curve in 5-space may be chosen as the third directrix curve of  $S_3$ , provided merely that the tangents to the three curves at corresponding points do not lie in a 4-space. This selection is always possible since three lines in 5-space do not in general lie in a 4-space.

Suppose now that we have given a non-developable ruled surface  $S$  which is determined by two directrix curves  $C_1$  and  $C_2$  of a three spread  $S_3$  defined by the system of equations (1). The transformation (2) merely changes the parametric representation of the directrix curves and has no effect either on  $S_3$  or  $S$ ; but the transformation (3) shifts the directrix curves on  $S_3$  from  $C_1, C_2, C_3$  into three new curves  $\bar{C}_1, \bar{C}_2, \bar{C}_3$  in such a way that  $\bar{C}_1$  and  $\bar{C}_2$  are on  $S$  if and only if  $\alpha_{13} = \alpha_{23} = 0$ . Thus the most general transformation (3) which leaves  $S$  unchanged is given by the equations

$$\begin{aligned} y_1 &= \alpha_{11}\bar{y}_1 + \alpha_{12}\bar{y}_2, \\ (4) \quad y_2 &= \alpha_{21}\bar{y}_1 + \alpha_{22}\bar{y}_2, \\ y_3 &= \alpha_{31}\bar{y}_1 + \alpha_{32}\bar{y}_2 + \alpha_{33}\bar{y}_3, \end{aligned} \quad \alpha_{33} \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \neq 0.$$

Inasmuch as  $S$  is independent of the nature of  $C_3$ , there is no loss of generality in putting  $\alpha_{31} = \alpha_{32} = 0$ , and in selecting for  $C_3$  a straight line, so that the last equation of (1) takes the simple form

$$y_3'' + 2p_{33}y_3' + q_{33}y_3 = 0.$$

It is possible to study the projective differential properties of any non-developable ruled surface in 5-space by means of the system of equations

$$(5) \quad \begin{aligned} y_1'' + 2p_{11}y_1' + 2p_{12}y_2' + 2p_{13}y_3' + q_{11}y_1 + q_{12}y_2 + q_{13}y_3 &= 0, \\ y_2'' + 2p_{21}y_1' + 2p_{22}y_2' + 2p_{23}y_3' + q_{21}y_1 + q_{22}y_2 + q_{23}y_3 &= 0, \\ y_3'' &+ 2p_{33}y_3' + q_{33}y_3 = 0, \end{aligned}$$

under the transformations (2) and

$$(6) \quad \begin{aligned} y_1 &= \alpha_{11}\bar{y}_1 + \alpha_{12}\bar{y}_2, \\ y_2 &= \alpha_{21}\bar{y}_1 + \alpha_{22}\bar{y}_2, \\ y_3 &= \alpha_{33}\bar{y}_3, \end{aligned} \quad \alpha_{33} \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \neq 0.$$

We shall frequently have occasion to make use of the result of transforming (5) by (6). If the new coefficients are denoted by  $\bar{p}_{ij}$  and  $\bar{q}_{ij}$ , we find by direct substitution that

$$(7) \quad \begin{aligned} \Delta \bar{p}_{ij} &= \sum_{k=1}^2 A_{ki} \left[ \alpha'_{kj} + \sum_{l=1}^2 p_{kl} \alpha_{lj} \right], \\ \Delta \bar{q}_{ij} &= \sum_{k=1}^2 A_{ki} \left[ \alpha''_{kj} + \sum_{l=1}^2 (2p_{kl} \alpha'_{lj} + q_{kl} \alpha_{lj}) \right], \\ \Delta \bar{p}_{i3} &= \alpha_{33}' \sum_{k=1}^2 A_{ki} p_{k3}, \\ \Delta \bar{q}_{i3} &= \alpha_{33}' \sum_{k=1}^2 2A_{ki} p_{k3} + \alpha_{33} \sum_{k=1}^2 A_{ki} q_{k3} \quad (i, j = 1, 2), \\ \alpha_{33} \bar{p}_{33} &= \alpha_{33}' + \alpha_{33} p_{33}, \\ \alpha_{33} \bar{q}_{33} &= \alpha_{33}'' + 2\alpha_{33}' p_{33} + \alpha_{33} q_{33}, \end{aligned}$$

where  $A_{ij}$  denotes the algebraic minor of  $\alpha_{ij}$  in the determinant

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}.$$

Again, the transformation (2) changes  $p_{ij}$  and  $q_{ij}$  into  $\bar{p}_{ij}$  and  $\bar{q}_{ij}$ , where

$$(8) \quad \begin{aligned} \bar{p}_{ii} &= \frac{1}{\xi'} (p_{ii} + \frac{1}{2} \eta), \\ \bar{p}_{ij} &= \frac{1}{\xi'} p_{ij} \quad (i \neq j), \\ \bar{q}_{ij} &= \frac{1}{(\xi')^2} q_{ij} \quad (i, j = 1, 2, 3), \end{aligned}$$

with

$$\eta = \frac{\xi''}{\xi'}.$$

The surface  $S$  is singular if and only if each set of three consecutive generators lies in a 4-space, that is, if and only if there exists a linear relation between  $y_1, y_2, y_1', y_2', y_1'', y_2''$ . The first two equations of (5) show that such a relation exists if and only if

$$(9) \quad p_{13}q_{23} - p_{23}q_{13} = 0.$$

From (7) and (8) it is evident that (9) is an invariant relation. Moreover, we see from (7) that it is always possible if (9) is satisfied, to transform (5) in such a way that in the new system either  $p_{13}=q_{13}=0$  or  $p_{23}=q_{23}=0$ . We shall in our future study of singular ruled surfaces assume that the equations (5) are of the simple form\*

$$(10) \quad \begin{aligned} & y_1'' + 2p_{11}y_1' + 2p_{12}y_2' + q_{11}y_1 + q_{12}y_2 = 0, \\ & y_2'' + 2p_{21}y_1' + 2p_{22}y_2' + 2p_{23}y_3' + q_{21}y_1 + q_{22}y_2 + q_{23}y_3 = 0, \\ & y_3'' + 2p_{33}y_3' + q_{33}y_3 = 0. \end{aligned}$$

It is possible for the invariant conditions  $p_{13}=q_{13}=p_{23}=q_{23}=0$  to be satisfied. In this case the first two equations of (5) or (10) show that the surface lies entirely in a 3-space. The resulting system has already been used by Wilczynski for the study of such ruled surfaces. Consequently, we shall assume that in equations (10)  $p_{23}$  and  $q_{23}$  do not both vanish identically.

A reference to equations (7) shows that the conditions  $p_{13}=q_{13}=0$  are left undisturbed by the transformation (6) of the dependent variables if and only if  $\alpha_{12}=0$ . Therefore in our study of singular surfaces we shall limit this transformation to the form

$$(11) \quad \begin{aligned} y_1 &= \alpha_{11}\bar{y}_1, \\ y_2 &= \alpha_{21}\bar{y}_1 + \alpha_{22}\bar{y}_2, \\ y_3 &= \alpha_{33}\bar{y}_3. \end{aligned}$$

The conditions  $p_{13}=q_{13}=0$  are undisturbed by the most general transformation (2) of the independent variable.

\* The variable  $y_3$  and its derivatives can of course be eliminated from (10). The form (10) has, however, the advantage in symmetry and in the fact that the same system may be used even if the surface lies wholly in a 4-space by simply imposing an invariant condition on the coefficients.

## 2. THE TRANSVERSAL SURFACE AND ASSOCIATED CURVES

The analytical conditions  $p_{13}=q_{13}=0$  which have been imposed in order to obtain (10) evidently determine for  $C_1$  a unique curve on the singular surface  $S$ . We shall denote it by  $C$ . The first equation of (10) shows that the osculating plane of  $C$  at a point  $P$  lies in the 3-space determined by the generator of  $S$  through  $P$  and the next consecutive generator of  $S$ . In other words,  $C$  is the unique curve at each point of which there exists a tangent to  $S$  intersecting three consecutive generators of  $S$ . It is evident geometrically that there can be only one such curve  $C$  unless the surface lies entirely in a 3-space, in which case it is true for all curves on the surface.

Let  $P$  be the point of intersection of  $C$  and the generator corresponding to  $x=x_0$  and let  $y_1+\alpha(x)y_2$  be the curve on the surface at  $P$  to which the line intersecting three consecutive generators is tangent. A point on the plane osculating this curve at  $P$  is given by  $y_1''+\alpha y_2''+2\alpha' y_2'+\alpha'' y_2$ , which point is seen by (10) to be on the tangent plane to  $S$  at  $P$  if and only if  $\alpha(x_0)=0$  and  $\alpha'(x_0)=p_{12}(x_0)$ . Thus a point on the tangent intersecting three consecutive generators is given by

$$(12) \quad \tau = y_1' + p_{12}y_2.$$

As  $x$  varies the line determined by the points  $y_1$  and  $\tau$  generates a ruled surface  $T$ , the *transversal surface*\* of  $S$ . The curve  $C$  is the intersection of  $S$  and  $T$ .

If the invariant condition  $p_{12}=0$  is satisfied,  $\tau=y_1'$ . We thus have the following theorem:

*The identical vanishing of the invariant  $p_{12}$  is the necessary and sufficient condition that a singular ruled surface  $S$  in 5-space possess a curve which is asymptotic in the ordinary sense and that the transversal surface associated with  $S$  be developable.*

It is easily verified directly that  $u_{12}=p_{12}'-q_{12}+p_{11}p_{12}+p_{12}p_{22}$  is an invariant. Since  $p_{12}=u_{12}=0$  is equivalent to  $p_{12}=q_{12}=0$  there follows immediately from equation (10) the additional theorem

*A singular ruled surface in 5-space possesses a straight line directrix if and only if the invariants  $p_{12}$  and  $u_{12}$  vanish simultaneously.*

Ranum† uses an equation of nearly the same form as the first equation of (10) for the classification of singular ruled surfaces in 4-space and 5-space.

\* Cf. Bompiani, loc. cit., p. 308.

† Cf. Ranum, American Journal of Mathematics, vol. 37 (1915), pp. 139-146 and 150-151.

The classification depends upon the vanishing of the coefficients corresponding to  $p_{12}$  and  $q_{12}$ .

The surface  $T$ , when not a developable, has associated with it a system of equations of the form of (1) in terms of the variables  $y_1, \tau, y_3$ . Assuming, accordingly, that  $p_{12} \neq 0$ , we may obtain one of its equations by eliminating  $y_2$  and  $y_2'$  from the first equation of (10) by means of (12). It is

$$(13) \quad y_1'' - \frac{2}{p_{12}} (p_{12}' - \frac{1}{2} q_{12} + p_{11}p_{12})y_1' - 2\tau' - q_{11}y_1 + \frac{1}{p_{12}} (2p_{12}' - q_{12})\tau = 0.$$

This equation shows at once that  $T$  is also singular and that  $C$  is its curve of intersection with its transversal surface. Moreover, since the expression for  $T$  corresponding to  $\tau$  for  $S$  is

$$(14) \quad y_1' - \tau = -p_{12}y_2,$$

the transversal surface of  $T$  must be  $S$ . Thus we have a theorem\* which holds for all singular ruled surfaces in 5-space, including all ruled surfaces in 4-space, whose transversal surfaces are not developables:

*The relation between a singular ruled surface and its transversal surface is reciprocal.*

Let us designate by  $d = y_1' + \alpha y_1 + \beta y_2$  a point on the tangent plane to  $S$  at  $y_1$ . As  $x$  varies,  $d$  will generate a curve  $D$ . On differentiating twice we find that the plane osculating  $D$  at  $d$  is the tangent plane to  $S$  at  $y$  if

$$\alpha = \frac{1}{2p_{12}} (2p_{12}' - q_{12} + 4p_{11}p_{12}), \quad \beta = 2p_{12}.$$

Consequently, the tangent planes osculate the curve  $D$  given by

$$(15) \quad d = y_1' + \frac{1}{2p_{12}} (2p_{12}' - q_{12} + 4p_{11}p_{12})y_1 + 2p_{12}y_2,$$

or

$$(15') \quad d = \tau + \frac{1}{2p_{12}} (2p_{12}' - q_{12} + 4p_{11}p_{12})y_1 + p_{12}y_2.$$

The line joining  $y_1$  and  $d$  intersects the line determined by  $\tau$  and  $y_2$  at  $\tau + p_{12}y_2$ . Since  $y_1' = \tau - p_{12}y_2$ , we have the following theorem:†

\* This theorem is well known for surfaces in 4-space. Cf. Bompiani, loc. cit.

† Ranum has proved this theorem for surfaces in 4-space. Cf. these Transactions, vol. 16 (1915), p. 92.

The line determined by  $y_1$  and  $d$  is the harmonic conjugate of the tangent to  $C$  at  $y_1$  with respect to the generator of  $S$  through  $y_1$  and the corresponding generator of  $T$ .

It is easy to obtain an infinite set of pairs of surfaces and transversal surfaces which have the same curves  $C$  and  $D$  and for which the above theorem holds. In order to show this fact let us put

$$e = d + ky_1', \quad f = d - ky_1',$$

where  $k$  is a constant. It is evident that the lines  $y_1e$  and  $y_1f$  are harmonic with respect to the lines  $y_1d$  and  $y_1y_1'$ . The ruled surface generated by  $y_1e$  has for one of its equations

$$2p_{12}ky_1'' - k(2p_{12}' - q_{12})y_1' - 2p_{12}e' + [2p_{12}\alpha' - 2p_{12}q_{11} - \alpha(2p_{12}' - q_{12})]y_1 + (2p_{12}' - q_{12})e = 0,$$

where

$$\alpha = \frac{1}{2p_{12}}(2p_{12}' - q_{12} + 4p_{11}p_{12}).$$

Thus, it is seen to be a singular surface which intersects its transversal surface in  $C$ . Since a point on the generator of its transversal surface is given by  $-f/2k$  and its point corresponding to  $d$  for  $S$  by  $-d/k$ , our statement is proved. This condition is the exact analogy of the relation between an infinite set of ruled surfaces in 3-space which have the same flecnode curve.\*

### 3. SURFACES IN 4-SPACE

Our ruled surface  $S$  lies entirely in a 4-space if every set of four consecutive generators is contained in a 4-space. Consequently, it must be possible in this case to express  $y_1^{(3)}$  and  $y_2^{(3)}$  in terms of  $y_1'', y_2'', y_1', y_2', y_1, y_2$ . It is always possible thus to express  $y_1^{(3)}$  but  $y_3'', y_3', y_3$  can be eliminated from the equation resulting from the differentiation of the second equation of (10) only if

$$(16) \quad \varphi_4 = 2(p_{23}'q_{23} - p_{23}q_{23}') + 4p_{23}(p_{23}q_{33} - q_{23}p_{33}) + q_{23}^2 = 0.$$

It is evident from its geometrical significance that the expression (16) is a relative invariant, a fact which may also be verified directly by means of equations (7) and (8).

If  $S$  lies in a 4-space but not in a 3-space,  $p_{23} \neq 0$ , a fact which is evident from (16) when it is remembered that  $p_{23} = q_{23} = 0$  are the conditions that  $S$

\* Cf. Wilczynski, loc. cit., p. 233.

lie entirely in a 3-space. In fact, if  $S$  is not in a 3-space, it is always possible to select  $C_3$  in such a way that  $p_{23} \neq 0$  regardless of whether  $S$  is in a 4-space or 5-space. The second equation of (10) shows that it is only necessary to choose the point on  $C_3$  corresponding to  $y_1$  and  $y_2$  on  $C$  and  $C_2$  in such a way that it is not in the 4-space determined by the tangent to  $C$  at  $y_1$  and the osculating plane to  $C_2$  at  $y_2$ .

The transformation of  $q_{23}$  as given by equations (7) shows that we may obtain a set of equations in which  $q_{23}$  vanishes by selecting  $\alpha_{33}$  to satisfy the equation  $2\alpha'_{33}p_{23} + \alpha_{33}q_{23} = 0$ . This choice of  $\alpha_{33}$  is always possible when  $p_{23} \neq 0$ . Moreover, the relation  $q_{23} = 0$  is maintained if  $\alpha_{33} = \text{const}$ . When  $p_{23} \neq 0$  and  $q_{23} = 0$  the relation (16) reduces to  $q_{33} = 0$ . We thus have the following theorem:

*The necessary and sufficient condition that  $S$  lie entirely in 4-space is  $\varphi_4 = 0$ . If  $S$  does not lie in a 3-space, this invariant relation may be always reduced to  $q_{33} = 0$  by the transformation of (10) into a system for which  $q_{23} = 0$ .*

Equations (10) with  $q_{23} = q_{33} = 0$  may be used to show analytically that there are  $\infty^2$  curves on a ruled surface in 4-space whose osculating 3-space at a given point contains the tangent plane to  $S$  at the point.\* Differentiation of the second equation of (10) shows that with  $q_{23} = q_{33} = 0$  we can express  $y_1$  in terms of  $y_2^{(3)}$ ,  $y_2''$ ,  $y_2'$ ,  $y_2$  if and only if

$$M = p_{23}(2p'_{21} + q_{21} - 4p_{11}p_{21}) - p_{21}(2p'_{23} - 4p_{23}p_{33}) = 0.$$

If in the transformation (11) we put  $\alpha_{11} = \alpha_{22} = \alpha_{33} = 1$ , equations (7) show that  $\alpha_{21}$  must satisfy a differential equation of the second order, in order that the transformed form of  $M$  shall equal zero. Our statement above is thus proved.

#### 4. THE CURVE OF INTERSECTION OF $S$ AND $T$

The differential equation for  $C$  is of the sixth order in  $y_1$ . By taking successive derivatives of the first equation of (10) and eliminating  $y_2$ ,  $y_3$  and their derivatives we have for the desired equation

$$(17) \quad \begin{vmatrix} y_1^{(6)} + l_{16}y_1' + m_{16}y_1 & l_{26} & l_{36} & m_{26} & m_{36} \\ y_1^{(5)} + l_{15}y_1' + m_{15}y_1 & l_{25} & l_{35} & m_{25} & m_{35} \\ y_1^{(4)} + l_{14}y_1' + m_{14}y_1 & l_{24} & l_{34} & m_{24} & m_{34} \\ y_1^{(3)} + l_{13}y_1' + m_{13}y_1 & l_{23} & l_{33} & m_{23} & m_{33} \\ y_1'' + 2p_{11}y_1' + q_{11}y_1 & 2p_{12} & 0 & q_{12} & 0 \end{vmatrix} = 0,$$

\* Cf. Bompiani, loc. cit., pp. 310-311.

where

$$(18) \quad \begin{aligned} l_{ij} &= l'_{i,j-1} + m_{i,j-1} - 2 \sum_{k=1}^3 p_{ki} l_{k,j-1}, \\ m_{ij} &= m'_{i,j-1} - \sum_{k=1}^3 q_{ki} l_{k,j-1} \end{aligned} \quad (j = 3, 4, 5, 6),$$

with

$$l_{i2} = 2p_{1i}, \quad m_{i2} = q_{1i}.$$

Equation (17) may be used for the study of the various projective differential properties of the curve  $C$ . In particular, we shall use it for the determination of the conditions under which  $C$  is located in the various sub-spaces.

(a)  $C$  a straight line. It has already been seen that the vanishing of the invariants  $p_{12}$  and  $u_{12}$  is the necessary and sufficient condition. This fact is also evident directly from (17).

(b)  $C$  in a 2-space but not a straight line. Since  $p_{12}$  and  $q_{12}$  cannot both vanish identically,  $l_{33} = m_{33} = l_{23} q_{12} - 2p_{12} m_{23} = 0$ . It follows at once from (18) that  $p_{23} = q_{23} = 0$ . Therefore,  $S$  must be in a 3-space and the invariant

$$\Delta_2 = l_{23} q_{12} - 2p_{12} m_{23} = 2(p'_{12} q_{12} - p_{12} q'_{12}) + 4p_{12}(p_{12} q_{22} - p_{22} q_{12}) + q_{12}^2$$

must vanish identically. The fact that  $\Delta_2$  is an invariant is evident geometrically and may be verified directly.

(c)  $C$  in a 3-space but not in a 2-space. The fact that  $p_{12}$  and  $q_{12}$  cannot both vanish shows that

$$l_{34} m_{33} - l_{33} m_{34} = 4p_{12}^2 \varphi_4 = 0.$$

Thus, either  $\varphi_4 = 0$  and  $S$  is in a 4-space, or  $p_{12} = 0$  and  $T$  is a developable. In the latter case the two additional equations which must be satisfied reduce immediately to  $2p_{23} q_{12}^3 = 0$  and  $q_{23} q_{12}^3 = 0$ , and the surface  $S$  is therefore in a 3-space.

If  $\varphi_4 = 0$  and  $p_{12} \neq 0$ , we must have also

$$\Delta_3 = \begin{vmatrix} l_{24} & l_{34} & m_{24} \\ l_{23} & l_{33} & m_{23} \\ 2p_{12} & 0 & q_{12} \end{vmatrix} = 0, \quad \Delta'_3 = \begin{vmatrix} l_{24} & m_{24} & m_{34} \\ l_{23} & m_{23} & m_{33} \\ 2p_{12} & q_{12} & 0 \end{vmatrix} = 0.$$

But

$$2p_{23} \Delta'_3 = -q_{23} \Delta_3 + 2p_{12} \varphi_4 (l_{23} q_{12} - 2m_{23} p_{12})$$

and therefore  $\Delta'_3$  vanishes as a consequence of the vanishing of  $\Delta_3$  and  $\varphi_4$ . It is easily verified by direct substitutions that  $\Delta_3$  is an invariant.

(d) *C* in a 4-space but not in a 3-space. It is necessary and sufficient that the determinant

$$\begin{vmatrix} l_{25} & l_{35} & m_{25} & m_{35} \\ l_{24} & l_{34} & m_{24} & m_{34} \\ l_{23} & l_{33} & m_{23} & m_{33} \\ 2p_{12} & 0 & q_{12} & 0 \end{vmatrix}$$

shall vanish. It is geometrically evident that the determinant must be an invariant. If the transversal surface is developable,  $p_{12}=0$  and the above determinant reduces to  $q_{12}^2 \varphi_4$ . It follows that in this case the surface *S* lies in a 4-space.

### 5. SEMI-CANONICAL FORMS OF THE EQUATIONS

Equations (7) show that the transformation (11) converts the coefficients  $p_{11}, p_{21}, p_{22}, p_{33}$  into  $\bar{p}_{11}, \bar{p}_{21}, \bar{p}_{22}, \bar{p}_{33}$  where

$$\begin{aligned} \Delta \bar{p}_{11} &= \alpha_{22}(\alpha'_{11} + \alpha_{11}p_{11} + \alpha_{21}p_{12}), \\ \Delta \bar{p}_{21} &= \alpha_{11}(\alpha'_{21} + \alpha_{11}p_{21} + \alpha_{21}p_{22}) - \alpha_{21}(\alpha'_{11} + \alpha_{11}p_{11} + \alpha_{21}p_{12}), \\ \Delta \bar{p}_{22} &= \alpha_{11}\alpha'_{22} + \alpha_{11}\alpha_{22}p_{22} - \alpha_{21}\alpha_{22}p_{12}, \\ \alpha_{33}\bar{p}_{33} &= \alpha'_{33} + \alpha_{33}p_{33}. \end{aligned} \quad (19)$$

If  $\alpha_{11}$  and  $\alpha_{21}$  are selected as a pair of solutions of the equations

$$\begin{aligned} \alpha'_{11} + \alpha_{11}p_{11} + \alpha_{21}p_{12} &= 0, \\ \alpha'_{21} + \alpha_{11}p_{21} + \alpha_{21}p_{22} &= 0, \end{aligned}$$

the transformed coefficients  $\bar{p}_{11}$  and  $\bar{p}_{21}$  become zero. Furthermore, if the value of  $\alpha_{11}$  and  $\alpha_{21}$  thus determined are substituted in the equation

$$\alpha_{11}\alpha'_{22} + \alpha_{11}\alpha_{22}p_{22} - \alpha_{21}\alpha_{22}p_{12} = 0$$

and  $\alpha_{22}$  is selected as a solution of the resulting equation, the coefficient  $\bar{p}_{22}$  also becomes zero. Finally,  $\bar{p}_{33}$  becomes zero if  $\alpha_{33}$  is selected to satisfy the equation  $\alpha'_{33} + \alpha_{33}p_{33} = 0$ .

Equations (10) may thus always be transformed into the simpler form

$$\begin{aligned} y_1'' + 2p_{12}y_2' + q_{11}y_1 + q_{12}y_2 &= 0, \\ y_2'' + 2p_{23}y_3' + q_{21}y_1 + q_{22}y_2 + q_{23}y_3 &= 0, \\ y_3'' + q_{33}y_3 &= 0. \end{aligned} \quad (20)$$

In accordance with the terminology of Wilczynski, we shall call (20) the *semi-canonical form of equations* (10).

We have already seen that it is always possible to select  $\alpha_{33}$  in such a way as to make  $q_{23}$  vanish if  $p_{23} \neq 0$ . Consequently, we have a second semi-canonical form in which  $q_{23}$  vanishes in place of  $p_{33}$ .

## 6. INVARIANTS AND COVARIANTS

Clearly every invariant and covariant of the general system (1) under the transformations (2) and (3) either vanishes identically or is an invariant or covariant of the special system (10) under the transformations (2) and (11). Moreover, a comparison of the form of equations (7) and (8) with the corresponding transformations of the coefficients of the system similar to (1), but with  $i=1, 2$ , shows that every invariant and covariant of the latter system either vanishes identically, or represents also an invariant or covariant of (10). Complete systems of invariants and covariants have been calculated\* for the system (1) for any value of  $i$  under the most general transformations which leave such a system invariant in form.

Making use of these facts and of equations (7) and (8), and remembering the invariants already discovered, we see easily that the following expressions are relative invariants:

$$\begin{aligned} & p_{12}, \quad u_{12}, \quad p_{23}, \\ & \theta_2 = (u_{11} - u_{22})^2 + 4u_{12}u_{21}, \\ & \theta_5 = 8\theta_2''\theta_2 - 9(\theta_2')^2 + 32(u_{11} + u_{22})\theta_2^2, \\ (21) \quad \varphi_4 &= 2(p_{23}'q_{23} - p_{23}q_{23}') + 4p_{23}(p_{23}q_{33} - p_{33}q_{23}) + q_{23}^2, \\ \Delta_2 &= l_{23}q_{12} - 2p_{12}m_{23} = 2(p_{12}'q_{12} - q_{12}'p_{12}) + 4p_{12}(p_{12}q_{22} - q_{12}p_{22}) + q_{12}^2, \\ \Delta_3 &= \begin{vmatrix} l_{24} & l_{34} & m_{24} \\ l_{23} & l_{33} & m_{23} \\ 2p_{12} & 0 & q_{12} \end{vmatrix}, \end{aligned}$$

where

$$(22) \quad u_{ik} = p_{ik}' - q_{ik} + \sum_{j=1}^2 p_{ij}p_{jk} \quad (i, k = 1, 2),$$

and where  $l_{ik}$  and  $m_{ik}$  are defined by equations (18).

It is easy to verify that the above invariants are a complete system provided merely that certain of them are not zero. In order to do this let us

\* Cf. *Invariants and covariants*.

assume that (10) has been transformed into a semi-canonical form so that  $p_{11} = p_{22} = p_{21} = 0$  and either  $p_{33} = 0$  or  $q_{23} = 0$ . Then if the invariants (21) are known functions of  $x$ , with  $p_{12}$ ,  $p_{23}$ ,  $\theta_2$ ,  $u_{12}$ ,  $\Delta_2$  not equal to zero, it is possible to solve for the coefficients of the semi-canonical form. The coefficients  $p_{12}$ ,  $p_{23}$ ,  $q_{12} = p_{12} - u_{12}$  are known at once. The expression for  $q_{22}$  follows immediately from  $\Delta_2$ . Since, in either semi-canonical form  $u_{11} + u_{22} = -q_{11} - q_{22}$  and  $u_{21} = -q_{21}$ , the expression for  $q_{11}$  can be obtained from  $\partial_5$  and then the expression for  $q_{21}$  from  $\theta_2$ . Since  $q_{33}$  does not appear in  $\Delta_3$  and since  $p_{23}$ ,  $q_{23}$  occur only in  $l_{34}$  which, with  $p_{11} = p_{22} = p_{21} = 0$ , reduces to

$$-2p_{23}(2p'_{12} + q_{12}) + 8p_{12}p_{23}p_{33} - 4p_{12}p'_{23} - 4p'_{12}p_{23} - 2p_{12}q_{23},$$

it is possible to obtain the expression for that one of the two coefficients  $p_{33}$  and  $q_{23}$  which has not been made zero. Finally,  $\varphi_4$  gives the expression for  $q_{33}$ .

Since higher derivatives of  $y_i$  than the second can be eliminated by means of equations (10) there can be only six independent covariants in addition to the complete system of invariants. Two of these are evidently  $y_1$  and  $y_3$ . Three others may be obtained immediately from the complete set for the system of equations of form (1) with two independent variables:\*

$$(23) \quad R_2 = \begin{vmatrix} y_1 & y_2 \\ r_{12} & r_{22} \end{vmatrix}, \quad T_{12} = \begin{vmatrix} y_1 & y_2 \\ t_{12} & t_{22} \end{vmatrix},$$

$$\psi_0 = 4T_{y2} + 2(u_{11} + u_{22})T_{12} + S_{12},$$

where

$$(24) \quad T_{y2} = \begin{vmatrix} t_{12} & t_{22} \\ r_{12} & r_{22} \end{vmatrix}, \quad S_{12} = \begin{vmatrix} y_1 & y_2 \\ s_{12} & s_{22} \end{vmatrix},$$

and

$$(25) \quad r_{i2} = \sum_{j=1}^2 u_{ij}y_j, \quad t_{i2} = y'_i + \sum_{j=1}^2 p_{ij}y_j, \quad s_{i2} = \sum_{j=1}^2 v_{ij}y_j \quad (i = 1, 2),$$

and

$$(26) \quad u_{ik} = p'_{ik} - q_{ik} + \sum_{j=1}^2 p_{ij}p_{jk}, \quad v_{ik} = u'_{ik} + \sum_{j=1}^2 (p_{ij}u_{jk} - p_{ik}u_{ji}) \quad (i, k = 1, 2).$$

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\* Cf. *Invariants and covariants*, pp. 346-352.

The remaining covariant may be selected from the set for the system(1) with three independent variables:

$$(27) \quad T_{23} = \begin{vmatrix} y_1 & y_2 & y_3 \\ r_{13} & r_{23} & r_{33} \\ t_{13} & t_{23} & t_{33} \end{vmatrix},$$

where

$$(28) \quad r_{i3} = \sum_{j=1}^3 u_{ij3} y_j, \quad t_{i3} = y_i' + \sum_{j=1}^3 p_{ij} y_j' \quad (i = 1, 2, 3),$$

and

$$(29) \quad u_{ik3} = p_{ik}' - q_{ik} + \sum_{j=1}^3 p_{ij} p_{jk} \quad (i, k = 1, 2, 3).$$

The independence of the above set of covariants is easily shown by means of the functional determinant formed with respect to the variables  $y_i, y_i'$  ( $i = 1, 2, 3$ ).

#### 7. DERIVATIVE SURFACES

The expressions  $t_{12}, t_{22}$  defined by (25) are cogredient with  $y_1, y_2$  under the transformation (11). The line joining the points  $t_{12}$  and  $t_{22}$  generates a surface  $S'$  as  $x$  varies and this surface we shall call the derivative surface of  $S$ . Since the transformation  $\xi = \xi(x)$  of the independent variable converts  $t_{12}$  and  $t_{22}$  into  $\bar{t}_{12}$  and  $\bar{t}_{22}$ , respectively, where

$$(30) \quad \bar{t}_{12} = \frac{1}{\xi'} (t_{12} + \frac{1}{2} \eta y_1), \quad \bar{t}_{22} = \frac{1}{\xi'} (t_{22} + \frac{1}{2} \eta y_2), \quad \eta = \frac{\xi''}{\xi'},$$

the surface  $S'$  changes as  $\eta$  changes. Since  $t_{12} = \tau + p_{11} y_1$ , the curve generated by  $t_{12}$  is always on the transversal surface  $T$ .

We proceed to find the differential equations of form (5) for  $S'$ . It is easily verified that

$$(31) \quad \begin{aligned} t_{12}' + p_{11} t_{12} + p_{12} t_{22} &= u_{11} y_1 + u_{12} y_2, \\ t_{22}' + p_{21} t_{12} + p_{22} t_{22} &= u_{21} y_1 + u_{22} y_2 - 2p_{23} y_3' - q_{23} y_3. \end{aligned}$$

By differentiating each equation of (31) and eliminating  $y_1$  and  $y_2$  and their derivatives, we obtain a set of equations of form (5) in the variables  $t_{12}, t_{22}, y_3$ . If in these equations we denote the coefficients corresponding to the coefficients  $p_{ij}$  and  $q_{ij}$  of (5) by  $P_{ij}$  and  $Q_{ij}$ , respectively, we find

$$\begin{aligned}
 IP_{11} &= \frac{1}{2} p_{11} I - \frac{1}{2} (v_{11} u_{22} - v_{12} u_{21} - p_{11} I), \\
 IP_{12} &= \frac{1}{2} p_{12} I + \frac{1}{2} A, \\
 IQ_{12} &= p_{12}' I - p_{12} (v_{11} u_{22} - v_{12} u_{21} - p_{11} I) - u_{12} I + p_{22} A, \\
 (32) \quad IP_{13} &= p_{23} A, \quad IQ_{13} = q_{23} A, \\
 IP_{23} &= p_{23} B + \frac{1}{2} (2p_{23}' + q_{23} - 4p_{23} p_{33}) I, \\
 IQ_{23} &= q_{23} B + (q_{23}' - 2p_{23} q_{33}) I,
 \end{aligned}$$

where

$$\begin{aligned}
 I &= u_{11} u_{22} - u_{12} u_{21}, \\
 (33) \quad A &= u_{12} v_{11} - u_{11} v_{12} + p_{12} I, \quad B = u_{12} v_{21} - u_{11} v_{22} + p_{22} I.
 \end{aligned}$$

Assuming\*  $I \neq 0$  we see at once that  $S'$  is singular if and only if

$$P_{23} Q_{13} - P_{13} Q_{23} = \frac{A \varphi_4}{2I} = 0.$$

Thus  $S'$  is singular if  $S$  is in a 4-space, a fact geometrically evident, or if  $A = 0$ . Equations (32) show that if  $A = 0$  the singular surface  $S'$  intersects its transversal in a curve on  $T$ . Now the transformations (11) leave  $A$  unchanged in form but the transformation  $\xi = \xi(x)$  converts  $A$  into  $\bar{A}$  where

$$\bar{A} = \frac{1}{(\xi')^5} [A + \frac{1}{2} p_{12} (u_{11} + u_{22}) \mu - \frac{1}{2} v_{12} \mu + \frac{1}{4} p_{12} \mu^2 + \frac{1}{2} u_{12} \mu'],$$

with

$$\mu = \eta' - \frac{1}{4} \eta^2.$$

We therefore have the following theorem:

*If  $S$  is not in a 4-space there are  $\infty^2$  derivative surfaces which are singular and each of them intersects its transversal surface in a curve on  $T$ ; if  $S$  is in a 4-space there are  $\infty^2$  of its singular derivative surfaces each of which intersects its transversal surface in a curve on  $T$ .*

Let us assume that the independent variable has been so chosen that  $A = 0$ . The point associated with  $S'$  which corresponds to  $d$  for  $S$  is given by

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\* Equations (31) show that  $I \neq 0$  if  $C_3$  is so selected as not to have a point in common with the tangent 3-space to  $S'$ .

$$\begin{aligned}
 (34) \quad t'_{12} + \frac{2P'_{12} - Q_{12} + 4P_{11}P_{12}}{2P_{12}} t_{12} + 2P_{12}t_{22} &= u_{11}y_1 + u_{12}y_2 + \frac{u_{12}}{p_{12}} t_{12} \\
 &= \frac{u_{12}}{p_{12}} d + \left( u_{11} - \frac{2p'_{12} - q_{12} + 2p_{11}p_{12}}{2p_{12}} \cdot \frac{u_{12}}{p_{12}} \right) y_1.
 \end{aligned}$$

Thus the point is on the line of intersection of the tangent planes to  $S$  at  $y_1$  and to  $S'$  at  $t_{12}$ , and also on the line joining  $y_1$  and  $d$ .

If the invariant  $u_{12}$  vanishes but  $C$  is not a straight line, that is,  $p_{12} \neq 0$ , we have  $A=0$  only if  $u_{11}=0$ . It follows from equation (34) that in this case every singular derivative surface which intersects its transversal surface in a curve on  $T$  is developable with its edge of regression on  $T$ .

Since  $2P_{12}=p_{12}$  when  $A=0$ , it follows that the transversal surface of a singular derivative surface which intersects its transversal surface in a curve on  $T$  is developable if and only if  $T$  is developable.

UNIVERSITY OF KANSAS,  
LAWRENCE, KANS.

## ON A TYPE OF COMPLETENESS CHARACTERIZING THE GENERAL LAWS FOR SEPARATION OF POINT-PAIRS\*

BY

C. H. LANGFORD

In a forthcoming paper by E. V. Huntington† a number of sets of postulates or determining conditions for the type of order called "separation of point-pairs" have been given. These sets are selected from a list of general properties which characterize reversible order on a closed line, and each of the sets is shown to imply all the others so that the several selections are equivalent. It is to be shown in the present paper that sets of postulates for separation of point-pairs are characterized by a property which is closely analogous to ordinary completeness. A class of propositional functions will be defined, to be called general laws,‡ to which any member of a set of postulates for this type of order belongs, and it will be shown that such sets are sufficient to determine the truth or falsity of any general law which can be constructed on the base  $K, R$ , the base for the set. This is a question of deducibility; one or the other of every pair of mutually contradictory general laws on  $K, R$  must be deducible.

The question of deducibility arises here in the following manner. It seems to be true from inductive considerations that each of these sets is a sufficient characterization of the type of order in question and thus that the theorems which follow from any one of them might be held to be exhaustive of the general properties which are understood to attach to systems involving separation of point-pairs. Any such set might then be taken as a set of defining properties for separation of point-pairs in the sense that any theorem which is commonly understood to hold for this type of order is implied by the postulates and no theorem which is recognized as not belonging to this type of order does follow from the postulates. In this sense

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† See Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 687-689. Also Bulletin of the American Mathematical Society, vol. 31 (1925), p. 405.

‡ Cf. E. V. Huntington, *Postulates for abstract geometry*, Mathematische Annalen, vol. 73 (1913), p. 528.

the set might be held to embody a satisfactory analysis of the current notion of reversible order on a closed line. On the other hand such a set might, of course, be said to define "separation of point-pairs" in the sense that the connotation of these words is thereby arbitrarily assigned. Now although the properties which are assigned by any one of Huntington's sets do characterize the type of order under consideration, it may be asked whether some essential property has not been overlooked in the sense that some theorem properly belonging to such systems is not implied by the postulates. Since the number of theorems which follow from sets of this sort is infinite, it would seem that the only way to answer such a question negatively would be to show that the set is *complete*. A complete set of postulates is a set which is such that any theorem which can be formulated in terms of the given base for the set is either implied by the postulates or else its contradictory is implied by them, that is, one or the other of every pair of mutually contradictory propositions follows from the set. Alternatively, a set is complete if any other set on the same base which implies it is implied by it, the sets in this instance being presumed to be self-consistent. In the case of sets of this kind no relevant theorem can be independent. But in the case of sets of general laws such as those for separation of point-pairs this property is certainly lacking; no categorical existence conditions are introduced and the cardinality of the system is left wholly undetermined.

It is proposed to show, however, that sets of general laws may be complete in a sense quite analogous to that of ordinary completeness and that any one of Huntington's sets does have this property. Since all of these sets are equivalent any one of them may be used, and for the present purpose the selection of postulates given below is most convenient. The system has for base a class  $K$  and a tetradic relation  $R(abcd)$  in terms of which the following properties are assigned.

00. For every  $a, b, c, d$  in  $K$ , if  $abcd$  is true then  $a, b, c, d$  are distinct.

F. For every distinct  $a, b, c, d$  in  $K$ , some permutation of  $a, b, c, d$  forms a true tetrad.

G. For every distinct  $a, b, c, d$  in  $K$ ,  $abcd \supset bcd a$ .

H. For every distinct  $a, b, c, d$  in  $K$ ,  $abcd \supset abdc$ .

R. For every distinct  $a, b, c, d$  in  $K$ ,  $abcd \supset dcba$ .

10. For every distinct  $a, b, c, d, x$  in  $K$ ,  $abcd \supset axcd$  or  $abcx$ .\*

It is to be noted that these propositional forms are all hypothetical in that they do not demand that some set of elements have the properties

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\*The condition that  $a, b, c, d$  be distinct in G, H, R, 10 is merely a matter of convenience since the hypotheses cannot be satisfied if they are not distinct.

given in their conclusions in order that the conditions demanded by the postulates be satisfied, but simply that the occurrence of such properties be contingent on the cardinality of  $K$ . This is of course true of all universal propositions. There is however a more important property which distinguishes sets like this one from other sets, and in order to bring out this distinction it will be necessary to discuss briefly certain well known types of proposition with regard to what may be termed their degree of quantification. Under degree of quantification we may have propositions which are singly quantified as contrasted with those which are multiply quantified.\* To use the simplest illustration, such propositions as "For some  $a, b, Rab$  holds" or "For every  $a, b, Rab$  holds" are singly-quantified propositions, whereas, "For every  $a$  some  $b$  is such that  $Rab$ " is a doubly quantified proposition. Or again, the proposition "Every element has an immediate successor" which occurs in connection with serial relations, when expanded is seen to be  $(a) : . (\exists b) : (c). ab \cdot \sim ac \vee \sim cb$ , which is a triply quantified proposition. Any proposition which involves the applicatives "some" or "every" is quantified, and every variable constituent of the propositional construct has some applicative which applies to it. The degree of quantification of a proposition is determined by the number of occurrences of the applicatives "some" and "every" in the quantifier of the proposition. A singly quantified proposition may be said to be about elementary propositions:  $(x) . \phi x$  is about  $\phi a$  and  $\phi b$ , etc. A doubly quantified proposition is, in this sense, about singly quantified propositions:  $(x) : (\exists y) . \phi(x, y)$  is about  $(\exists y) . \phi(a, y)$  and  $(\exists y) . \phi(b, y)$ , etc., and each of these singly quantified propositions is about elementary propositions. A triply quantified proposition is about doubly quantified propositions which are about singly quantified propositions, and so on.† In the current theory each variable is assigned a different scope and a separate applicative attaches to each variable.‡ Thus the proposition (1) For every  $x, y, \phi(x, y)$  would be written (2) For every  $x$  every  $y$  is such that  $\phi(x, y)$ , or (3) For every  $y$  every  $x$  is such that  $\phi(x, y)$ . (2) and (3) are clearly equivalent to (1) which differs from them in that its quantified constituents have the same scope, while in (2), for example,  $x$  has a wider scope than  $y$ . (2) and (3) are doubly quantified and are reducible to (1) which is singly quantified.

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\*The terms are due to Mr. W. E. Johnson, although his use of them differs slightly from the present one. See *Mind*, new ser., vol. 17, p. 240 ff.

† Compare a paper *Some theorems on deducibility*, forthcoming, in the *Annals of Mathematics*.

‡ See, however, *Principia Mathematica*, 2d edition, vol. 1, pp. xx-xxii.

Whenever two applicatives which are both universal or both particular are juxtaposed it is always possible to reduce the degree of quantification in this way, and it is important for our present purpose to effect this reduction, so that when a proposition is referred to as singly quantified it may have  $n$  variable constituents but they must all be affected by the same applicative. The present discussion is to be confined to first-order functions. A first-order function is a function whose values are first-order propositions. A first-order proposition is a proposition which contains variables denoting individuals but does not contain any variable functions.\* All of the foregoing postulates are first-order functions.

The point about singly quantified propositions is that when a proposition has the form "For every  $a, b, c, \dots, \phi(a, b, c, \dots)$ " or "For some  $a, b, c, \dots, \phi(a, b, c, \dots)$ " it refers separately to subclasses of  $n$  elements, so that if we know the relational structure of any such subclass we know whether the proposition is true or false. Now in the case of such expressions  $n$  will be finite, so that the relational structure of the subclass can be given extensionally and determinately, and this is important in connection with the subsequent proofs which are concerned solely with single quantifications.

In all of the postulates of the foregoing set the quantified constituents all have the same scope. These postulates are singly quantified. But in adding further properties to the list in order to determine a particular type of reversible order on a closed line we should certainly require other than singly quantified statements. For example we might wish to assign the property of *density*, "Any two points of the closed line are separated by some pair of points," and this would have to be written  $(a, b) : \cdot (\exists c, d) : a, b \in K. \supset c, d \in K \cdot Racbd$ , which is a doubly quantified postulate, and it is not reducible. In fact it will be shown presently that no further independent singly quantified hypothetical first-order functions can be added to the above list, and this is the sense in which the set is to be shown to be complete. By a general law, as used in the present discussion, we mean, then, any first-order functions which is hypothetical and singly quantified.

There are two ways in which the postulates of the set (00-10) may be classified both of which are important. Postulate 00 differs from the other members of the set in that it is concerned with  $R$ -tetrads in which not all of the elements are distinct. This is brought out by expressing 00 in the form "For every  $a, b, c, d$  in  $K$ , if  $a, b, c, d$  are not all distinct, then  $Rabcd$  fails." The other postulates have to do with  $R$ -tetrads in which all of the elements are

\*Cf. *Principia Mathematica*, 2d edition, vol. 1, p. xxiii.

distinct. It would of course be possible to divide 00 into three different statements having to do respectively with ordered tetrads of one, two, and three distinct elements, but this is unnecessary in the present instance since precisely the same assertion holds for each of these cases. If we consider the possible values of  $R$ , they fall into four classes according as the tetrads involve one, two, three, or four distinct elements, and these sets of values are non-overlapping. Consequently the validity-values\* assigned to any one of the sets are independent of those assigned to any other, and it follows that postulates which refer to different classes in this respect are wholly independent of one another. Since 00 refers solely to the first three sets of tetrads while  $F$ -10 refer to the fourth set alone, 00 is wholly detached† from the remaining postulates, and for this reason it may be treated separately. On the other hand, we may classify the members of the set (00-10) according to the number of distinct elements in the sets on whose  $R$ -structure they place restrictions. 00 places restrictions on the  $R$ -structure of every set of one, two, or three distinct elements;  $F$  places restrictions on the structure of every four distinct elements, and the same is true of  $G$ ,  $H$ , and  $R$ ; 10 restricts the  $R$ -structure of every set of five distinct elements.

Postulates  $F$ ,  $G$ ,  $H$ , and  $R$  are alike in that they are about ordered tetrads of distinct elements and involve reference to just four elements. If we consider any subclass of four elements, it is clear that the relation  $R$  may hold or fail independently for each of the twenty-four ordered tetrads which may be formed of these elements. The force of  $G$  may then be expressed by writing its conclusion in the form  $abcd \cdot \overline{bcda} = 0$ , that is,  $abcd \cdot \overline{bcda}$  cannot occur. Similarly, the conclusion of  $H$  may be written  $abcd \cdot \overline{abdc} = 0$ , while that of  $R$  will be  $abcd \cdot \overline{dcba} = 0$ . In the case of  $F$  we should have to write out all of the twenty-four ordered tetrads and exclude the case in which the relation  $R$  fails for all of them. The formulation of  $F$ ,  $G$ ,  $H$ , and  $R$  in this way shows that their force is to be found in the possibilities which they exclude. Different permutations of the same set of elements are distinguished relatively to one another and to the permutations of related sets, but in no sense absolutely. Accordingly, if a postulate is satisfied by a given selection of validity-values attaching to the ordered tetrads formed of  $a$ ,  $b$ ,  $c$ ,  $d$  then the postulate will be satisfied if any of the

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\*A tetrad  $abcd$  has a positive validity-value if  $R(abcd)$  is true and a negative validity-value if  $R(abcd)$  is false.

† Cf. E. V. Huntington, *A new set of postulates for betweenness*, these Transactions, vol. 26 (1924), p. 275.

elements  $a, b, c, d$  are interchanged. The twenty-four permutations of any four elements in  $K$  must be characterized by some distribution of positive and negative validity-values;  $H$  demands that no two permutations related as are  $abcd$  and  $abdc$  shall both be positive;  $F$  demands that not all permutations fail, while  $G$  and  $R$  demand that no two permutations related as are  $abcd$  and  $bcda$  or as are  $abcd$  and  $dcba$  shall be positive and negative respectively.

If we consider the different forms which arise by distributing positive and negative validity-values in different ways over the several ordered tetrads formed of four elements, the question arises as to how many of these forms are excluded by  $F, G, H$ , and  $R$ . If  $F, G, H, R$  should exclude all possible distributions, then they would be inconsistent with the existence of at least four elements in the class  $K$ . If they should be consistent with the occurrence of more than one type of distribution of validity-values, then further restricting conditions independent of  $F, G, H$ , and  $R$  are possible and  $F, G, H, R$  do not select a determinate form for any four elements. If, however, there is only one type of distribution of validity-values for the permutations of four elements in ordered tetrads of distinct elements which satisfies  $F, G, H$ , and  $R$ , then any further general law about four elements in ordered tetrads of distinct elements is either redundant with  $F, G, H, R$  or else, with its addition, the set implies that there are not at least four elements in  $K$ .

The consideration that  $F, G, H, R$  may exclude all possible forms of distribution of validity-values but one leads to an alternative procedure for formulating postulates by which it is possible to know that a determinate form is selected. Instead of attempting to determine when all forms but one have been excluded we may reverse this procedure and demand that any  $n$  elements have a specified form. We may express by  $R'(abcd)$  the holdure of  $R$  for the eight permutations which can be obtained by a cyclic permutation of  $abcd$  and its reverse.  $R'(abcd)$  means then  $Rabcd \cdot Rbcda \cdot Rdcba$ , etc. Also, when for some permutation of the elements  $x, y, z, w$ ,  $R$  holds, we may say that  $x, y, z$ , and  $w$  have the form  $R(ABCD)$ , which means, then, that  $Rxyzw$  or  $Ryzwx$  or, etc. If  $x, y, z, w, t$  can be so identified with  $A, B, C, D, E$  that  $R(ABCD) \cdot R(BCDE)$  holds, then  $x, y, z, w, t$  may be said to have the form  $R(ABCD) \cdot R(BCDE)$ . Similarly, if for some permutation of  $x, y, z, w$ ,  $R'$  holds, we may say that  $x, y, z, w$  have the form  $R'(ABCD)$ . Consider the postulate

- (1) Any four  $K$ -elements on tetrads of distinct elements have the form  
 (a)  $R'(ABCD)$ ; (b)  $R$  fails for all permutations other than those involved in (a).

This formulation is equivalent to the assertion that all forms of proposition other than this one are false; it ascribes a determinate form to any set of four elements on tetrads of distinct elements.\* Given any four  $K$ -elements  $a, b, c, d$ , then some permutation of  $a, b, c, d$  must have this character; what permutation that is is quite irrelevant and is in fact meaningless except in the case of a particular application. That this postulate determines completely the relational structure of  $a, b, c, d$  may be seen as follows. Consider the set of all possible ordered tetrads on  $a, b, c, d$  and let a set of validity-values be assigned in accordance with the postulate and regard the result as a compound proposition about the holdure and failure of  $R$  for the several ordered tetrads. The statement assigns a determinate validity-value to each tetrad of the set. Now consider any rearrangement of  $a, b, c, d$  in this proposition. This will give rise to a proposition which must have the same form as the original one and it too will include all possible permutations of  $a, b, c$ , and  $d$ . If in the second proposition the validity-value of some permutation of  $a, b, c, d$  differs from the validity-value of this permutation in the first proposition, then the two statements are incompatible. If, however, every permutation has the same validity-value in the two cases, then the second is precisely the same proposition as the first. It follows that no two propositions of this kind can both be true. The postulate therefore exhaustively characterizes, in terms of general law, any four elements on tetrads of distinct elements.

It will be shown that (1) is equivalent to  $F, G, H$ , and  $R$ . That  $F, G, H, R$  imply (1) may be seen as follows. Consider any four elements in  $K$ . By  $F$ , some permutation of these elements, say  $abcd$ , is such that  $Rabcd$  holds. Then by  $G$ ,  $bcd a, cdab, dabc$  hold, and by  $R$ ,  $dcba, adcb, badc, cbad$  hold. From the fact that the relation  $R$  holds for these eight tetrads it follows by  $H$  that  $R$  fails for all other permutations. But this gives the form demanded by (1).

That  $F, G, H, R$  follow from (1) is seen immediately by noting that any four elements having the form demanded in (1) satisfy  $F, G, H$ , and  $R$ , so that if every four elements in  $K$  have this form,  $F, G, H$ , and  $R$  must hold. This shows that  $F, G, H, R$  are complete within the domain to which they refer directly.

We now proceed to formulate, in terms of the notion of the form of a set of elements, a set of postulates for separation of point-pairs which will be shown to be equivalent to the set 00-10. The set has for base a class  $K$

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\*The complete characterization of a set of  $n$  elements by assigning a determinate set of validity-values is due to H. M. Sheffer. The procedure is applied here to values of functions.

and a tetradic relation  $R$  in terms of which the following properties are assigned.

1. Every element in  $K$  has the form  $\overline{AAAA}$ .
2. Every pair of elements in  $K$  have the form  $\overline{AABAB} \cdot \overline{AABA} \cdot \overline{ABAA} \cdot \overline{BAAA} \cdot \overline{AABB} \cdot \overline{ABAB} \cdot \overline{BAAB} \cdot \overline{ABBA} \cdot \overline{BABA} \cdot \overline{BBAA} \cdot \overline{ABBB} \cdot \overline{BABB} \cdot \overline{BBAB} \cdot \overline{BBBA}$ .
3. Every three elements in  $K$  have the form (a)  $\overline{AABC} \cdot \overline{AACB} \cdot \overline{ABAC} \cdot \overline{ACAB} \cdot \overline{ABCA} \cdot \overline{ACBA} \cdot \overline{BAAC} \cdot \overline{CAAB} \cdot \overline{BACA} \cdot \overline{CABA} \cdot \overline{BCAA} \cdot \overline{CBAA}$ ; (b) the tetrads obtained by interchanging  $A$  and  $B$  in (a) are all negative and those obtained by interchanging  $A$  and  $C$  in (a) are all negative.
4. Every four elements in  $K$  on tetrads of distinct elements have the form (a)  $R'(ABCD)$ , (b)  $R$  fails for every other permutation of these variables.
5. Every five elements in  $K$  on tetrads of distinct elements have the form (a)  $R'(ABCD) \cdot R'(ABCE) \cdot R'(ABDE) \cdot R'(ACDE) \cdot R'(BCDE)$ , (b)  $R$  fails for every permutation of any four of the variables  $A, B, C, D, E$  which is not asserted in (a).

It is clear that 1, 2, and 3 can be compounded into one postulate and that they are together equivalent to 00. It might seem on casual examination as if 4 follows from 5 and is thus superfluous, and this is indeed the case if there are not just four elements in  $K$ . But if the set 1-5 is to be equivalent to 00-10, this case cannot be excluded, so that 4 is necessary. The set 1-5 is not to be regarded as having any value in use; it is introduced here partly as a step in the proof of the completeness of the general laws 00-10 and partly for its theoretical interest.

THEOREM I. 00 implies 1, 2, and 3.

For  $\overline{abcd}$  if  $a, b, c, d$  are not all distinct.

THEOREM II.  $F, G, H$ , and  $R$  imply 4.

This theorem has already been proved. III and IV are preliminary to V, and are, of course, to be derived from 00-10.

THEOREM III.  $abxc \cdot abcy \supset abxy$  if  $a, b, c, x, y$  are all in  $K$ .

By 10,  $abxc \cdot y \supset ayxc$  or  $abxy$  and  $abcy \cdot x \supset axcy$  or  $abcx$ . But  $abcx$  and  $abxc$  are incompatible, by  $H$ . Hence  $axcy$ ;  $ayxc \supset yxca \supset xcay$ , by  $G$ . But  $axcy \supset xcya$ , contrary to  $xcay$ . Hence  $\overline{ayxc}$ , hence  $abxy$ .

THEOREM IV.  $abcx \cdot abcy \supset abxy$  or  $abyx$ , if  $a, b, c, x, y$  are in  $K$  and  $x \neq y$ .

By  $G$ ,  $abcx \supset bcxa$  and  $abcy \supset bcya$ ;  $bcxa \cdot y \supset byxa$  or  $bcxy$ , and  $bcya \cdot x \supset bxya$  or  $bcyx$ , by 10;  $bcyx$  and  $bcxy$  are contrary, by  $H$ ; hence  $byxa$  or  $bxya$ . But  $byxa \supset abyx$  and  $bxya \supset abxy$ .

THEOREM V.  $F, G, H, R$ , and 10 imply 5.

Consider any five elements in  $K$ . Any four of these elements must have some permutation, say  $abcd$ , such that  $abcd$  holds, by  $F$ . (1)  $abcd \cdot x \supset axcd$  or  $abxc$ , by 10. Suppose  $axcd$ ;  $axcd \supset xcda \supset cdax$  and  $abcd \supset bcda \supset cdab$ . (2)  $cdax \cdot cdab \supset cdbx$  or  $cdxb$ , by IV. Suppose  $cdbx$ . Then we have  $abcd \cdot axcd \cdot bxcd$ . Also,  $cdbx \cdot a \supset cabx$  or  $cdba$ , by 10, and  $abcd \supset bcda \supset cdab$ , by  $G$ . But  $cdab$  is contrary to  $cdba$ , by  $H$ . Hence  $cabx$ . Also,  $cabx \supset abxc$ , by  $G$ , and  $abxc \cdot abcd \supset abxd$ , by III. We have then  $abxc \cdot abxd \cdot abcd \cdot axcd \cdot bxcd$ .

Reverting to the alternative possibility in (2), suppose  $cdxb$ . We have then  $abcd \cdot axcd \cdot xbcd$ . Moreover,  $cdxb \cdot a \supset cabx$  or  $cdxa$ , and  $cdxa$  is contrary to  $cdax$ . Hence  $caxb$ . But  $caxb \supset axbc$ , and  $axbc \cdot axcd \supset axbd$ . Hence, we have  $axbc \cdot axbd \cdot axcd \cdot abcd \cdot xbcd$ .

Reverting to the alternative in (1), suppose  $abxc$ . (3)  $abxc \cdot abcd \supset abdx$  or  $abxd$ . If  $abdx$ ,  $abdx \cdot c \supset acdx$  or  $abdc$ . But  $abdc$  is contrary to  $abcd$ . Hence  $acdx$ . Also,  $abcd \supset bcda$ ;  $bcda \cdot x \supset bcdx$  or  $bxda$ ;  $bxda \supset abxd$ , contrary to  $abdx$ . Hence,  $abcd \cdot abxc \cdot abdx \cdot acdx \cdot bcdx$ .

Reverting to the alternative in (3), suppose  $abxd$ . We have  $abcd \cdot abxc \cdot abxd$ . And  $abxd \cdot c \supset acxd$  or  $abxc$ . But  $abxc$  is contrary to  $abxc$ , hence  $acxd$ . Moreover,  $acxd \supset cxda$  and  $abxc \supset cxab$ , and  $cxda \cdot cxab \supset cxdx$ , by IV. But  $cxdx \supset bcxd$ . Hence  $abxc \cdot abcd \cdot abxd \cdot acxd \cdot bcxd$ .

In any case, then, some permutation of any five distinct elements has this  $R$ -structure. But  $abxc \supset R'(abxc)$ . Similarly, we have  $R'(abcd)$ ,  $R'(abxd)$ ,  $R'(acxd)$ , and  $R'(bcxd)$ , and this is the form required in the theorem.

Theorems I, II, and V show that set 1-5 follows from set 00-10. The following theorems establish the converse relation.

THEOREM VI. 1, 2, and 3 imply 00.

1, 2, and 3 entail the failure of every ordered tetrad in which the elements are not all distinct and this is precisely the force of 00.

THEOREM VII. 4 implies *F*.

For if every four elements in *K* have the form demanded in 4 then not all permutations fail.

THEOREM VIII. 4 implies *G*.

An inspection of the form which any four elements must have in accordance with 4 shows that  $abcd \cdot \overline{bcda}$  cannot occur. Similarly,

THEOREM IX. 4 implies *H* and *R*.

THEOREM X. 5 implies 10.

For 10 requires that if  $abcd$  holds and  $x$  belongs to *K*, then  $\overline{axcd}$  and  $\overline{abcx}$  cannot both be true. If 5 holds every set of five distinct elements in *K* must have the form given in 5 and in this form the combination  $abcd \cdot \overline{axcd} \cdot \overline{abcx}$  does not occur.

This establishes the equivalence of sets 1-5 and 00-10, and we are now in a position to consider the question whether further independent general laws can be added to either of these sets, and of course if some law exists which is independent of one set it will be independent of the other also. It is a characteristic of the set 1-5 that each of the postulates is expressed in a completely expanded and extensional form. Postulate 4, for example, demands that every four elements in *K* shall exhibit the type of distribution of validity-values specified in that postulate, and the form there given includes all possible permutations of four elements in tetrads of distinct elements. Now consider any postulate, say *Y*, which is such that it refers to just four elements on tetrads of distinct elements—postulate *G*, for example. In general such a postulate will be satisfied by a set of four elements having certain distributions of validity-values and will not be satisfied by other distributions. If *Y* is satisfied by a set of elements having the particular distribution of values demanded by 4, then since 4 demands that every four elements in *K* have this form, *Y* will be implied by 4 as it must hold whenever 4 holds. On the other hand, if *Y* is not satisfied by a set of elements having the particular form demanded in 4, then 4 implies that *Y* is false.

This argument requires a slight modification and extension to include extreme cases. In the first place it may happen that *K* has less than four elements. In this case every general law of the kind is true, and 4 materially implies *Y*. Properly, of course, that *K* does not have at least four elements implies *Y*; and it may be well to obviate a possible source of confusion

here. When it is asserted that for every general law  $Y$ , 4 implies  $Y$  or else 4 implies that  $Y$  is false, this assertion is not to be understood to mean that for any  $Y$ , 4 implies that  $Y$  is true for every value of  $K$  or else 4 implies that  $Y$  is false for every value of  $K$ . It is to be understood to mean that for any value of  $K$ , 4 implies that  $Y$  is true or 4 implies that  $Y$  is false, and this again is, of course, to be distinguished from the triviality that for any value of  $K$ , every  $Y$  is such that 4 implies that  $Y$  is true or  $Y$  is false. For example, the general law  $abcd \supset abdc$  for every  $a, b, c, d$  in  $K$  is such that if  $K$  has at least four elements, Postulate 4 implies that it is false; whereas, if  $K$  does not have at least four elements 4 implies that it is true.

There is another special case which needs to be considered. It was said that in general  $Y$  will be satisfied by some distributions of validity-values on four elements and will not be satisfied by others. It may happen that  $Y$  is satisfied by every distribution of validity-values. In this case, however,  $Y$  follows from every general law and itself contributes no postulational information. On the other hand  $Y$  may be such that it is incompatible with the existence of at least four elements in  $K$ . Here, however, if  $K$  does have at least four elements then  $Y$  must be false and if  $K$  does not have at least four elements  $Y$  is true. Hence,

**THEOREM XI.** *Every general law  $Y$  on the base  $K, R$  which refers to just four elements in tetrads of distinct elements is such that 4 implies  $Y$  or else such that 4 implies that  $Y$  is false.*

Precisely similar considerations establish the corresponding theorem with respect to Postulate 5.

**THEOREM XII.** *Every general law  $Y$  on the base  $K, R$  which refers to just five elements on tetrads of distinct elements is such that 5 implies  $Y$  or such that 5 implies the contradictory of  $Y$ .*

Analogous theorems clearly follow for Postulates 1, 2, and 3. But it is also easy to see that here the theorems may be generalized immediately so as to include general laws referring to any finite number of elements. Postulate 1 demands that every element in  $K$  have the form  $\overline{AAAA}$ , and it follows from this that any  $n$  elements on tetrads of a single element have a determinate form. Hence any general law referring to tetrads of a single element which is such that it implies that at least one tetrad holds is incompatible with 1, whereas any such law which does not imply the holdure of at least one tetrad  $aaaa$  is implied by 1. Similar considerations hold for 2 and 3; 2 implies that for any  $n$  elements on ordered tetrads of two

distinct elements every tetrad fails, and 3 implies that any  $n$  elements on tetrads of three distinct elements are such that every tetrad fails. Hence,

**THEOREM XIII.** *Every general law on the base  $K, R$  which involves reference to  $n$  elements on ordered tetrads of elements not all distinct is such that it is implied by Postulates 1, 2, and 3 or else such that its contradictory is implied by 1, 2, and 3.*

It is not always the case that a postulate refers solely to tetrads of non-distinct elements or solely to tetrads of distinct elements; many restrict in some way the validity-values of both. In dealing with such a possibility it is necessary to consider the combined force of Theorems XI, XII, and XIII, and indeed we have already combined 1, 2, and 3 additively in arriving at XIII. Consider any four elements in  $K$  formed into all possible ordered tetrads of distinct and non-distinct elements. XIII requires that  $R$  fail for all tetrads of non-distinct elements, whereas XI prescribes a determinate form for those tetrads formed of distinct elements. Clearly, the total form exhibited by any four elements is a determinate one, and any general law whatever, a value of which involves not more than four elements, is dependent on XI and XIII. Similarly, XII and XIII imply that any five elements in  $K$  have a completely expanded form with a determinate distribution of validity-values. Hence,

**THEOREM XIV.** *Any general law which can be formulated in terms of  $K$  and  $R$  and whose values do not involve more than five elements is such that either it or its contradictory is implied by the set 1-5.*

It is certain, then, that if there exist any general laws independent of 1-5 they must have values which involve more than five elements. In order to show that such laws do not exist it will be necessary to show that as a consequence of 1-5 any  $n$  elements have a determinate form. Postulates 1, 2, and 3 have already been shown to include the case of  $n$  elements where  $n$  is any finite number. It remains to extend 5 in this way.

We may denote by  $C(abc \dots)$  the assertion of all those ordered tetrads which are obtained in the expression  $C$  by reading in the order from left to right. Thus  $C(abcde)$  means  $Rabcd \cdot Rabce \cdot Rbcde$ , etc.

**THEOREM XV.** *For every set of  $n$  elements in  $K$  ( $n > 3$  and finite) some permutation  $abc \dots$  is such that  $C(abc \dots)$  holds.*

The theorem has already been established for  $n \leq 5$  in XI and XII. Consider the hypothesis that the theorem holds for every subclass in  $K$  of just  $n$  elements, where  $n \geq 5$ .

Any subclass of  $K$  having  $n+1$  elements has  $n+1$  subclasses of  $n$  elements each, and for each such subclass the relation  $C$  holds for some permutation of its elements.

It will be shown that there are at least three elements  $a, b, c$  in the set of  $n+1$  elements such that  $abcx$  holds for every element  $x$  of the set other than  $a, b$ , and  $c$ .

Take any subclass of  $n$  elements. Some permutation  $C(abc \dots)$  holds. Call the element not belonging to this set of  $n$  elements  $q$ . Now  $abcx$  holds for every  $x$  other than  $a, b$ , and  $c$  in the expression  $C$ . If  $abcq$ , then the requirement is satisfied.

Let  $z$  be some one of the set of  $n+1$  elements other than  $a, b, c$ , and  $q$ . Then  $abcz$  holds. By 10,  $abcz \cdot q \supset abcq$  or  $aqcz$ . Suppose  $aqcz$ . We have  $abcz$ ;  $abcz \supset bcza \supset czab$ ;  $czab \cdot q \supset cqab$  or  $czaq$ . If  $cqab$ , then  $abcq$ , and the requirement is satisfied. If the condition is not to be satisfied  $czaq$  must be true.  $czaq \supset qazc \supset azcq$ , and  $abcz \supset bcza \supset azcb$ ;  $azcq \cdot azcb \supset azqb$  or  $azbq$ . Moreover,  $azqb \supset zqba \supset abqz$ , and  $azbq \supset zbqa \supset aqbz$ .

Suppose  $abqz$ . Then  $abqz \supset zqba \supset qbaz \supset bazq$ , and  $abcz \supset zcba \supset cbaz \supset bazc$ ;  $bazq \cdot bazc \supset baqc$  or  $bacq$ .

If  $baqc$ ,  $baqc \supset abcq$ . Here  $a, b$ , and  $c$  satisfy the condition.

If  $bacq$ ,  $bacq \supset abqc$ , and  $abqc \cdot abcx \supset abqx$ . Here  $a, b$ , and  $q$  satisfy the condition.

Suppose  $aqbz$ . Then  $aqbz \cdot c \supset acbz$  or  $aqbc$ . But  $abcz \supset zabc$  and  $acbz \supset zacb$ , contrary to  $zabc$ . Hence  $aqbc$ . Now  $aqbc \cdot x \supset axbc$  or  $aqbx$ . But  $axbc$  is contrary to  $xabc$ . Hence  $aqbx$ . Here  $a, q$ , and  $b$  satisfy the condition. Hence, for some  $a, b, c$ ,  $abcx$  for every  $x$  other than  $a, b$ , and  $c$  in the set of  $n+1$  elements.

Let  $a, b, c$  be three elements in the set such that  $abcx$  for every  $x$  other than  $a, b, c$ , and let the relation  $C$  hold for the set of  $n$  elements to which  $c$  does not belong.  $C(xyz \dots wt) \supset C(tw \dots zyx)$ , so that for one or the other of these expressions  $a$  and  $b$  occur in the order  $ab$ . Consider the expression for which this is the case.

There is no pair of elements  $x, y$  in  $C$  such that  $axby$ . For  $abcx \cdot abcy \supset abxy$  or  $abyx$ . If  $abxy$ ,  $abxy \supset yabx$  and  $axby \supset yaxb$ , contrary to  $yabx$ . If  $abyx$ ,  $abyx \supset xyba \supset ybax$  and  $axby \supset ybxa$ , contrary to  $ybax$ . Then  $axby$  is always false. Hence, in the expression  $C$  every element  $x$  is such that  $a, x$ , and  $b$  are in the order  $axb$  or no element is so.

We may permute the elements of  $C$  cyclically so as to bring  $a$  into the first place. We have then either  $C(a \dots x \dots b)$  or  $C(ab \dots x \dots)$ . If  $C(a \dots x \dots b)$ , then  $C(\dots x \dots ba)$  and this implies  $C(ab \dots x \dots)$ ;

so that in any case  $C$  can be so chosen that  $a$  and  $b$  fall in the first and second places respectively. We have then  $abxy$  for every  $x, y$  in the order  $xy$  in  $C$ .

$abcx \supset xcba \supset cbax$  and  $abcy \supset ycba \supset cbay$ ;  $cbax \cdot cbay \supset cbxy$  or  $cbyx$ ;  $cbxy \supset bxyx$  and  $abxy \supset bxya$ ;  $bxyx \cdot bxya \supset bxax$  or  $bxca$ ;  $bxax \supset xacb$ , contrary to  $xabc$ , and  $bxca \supset abxc$ , contrary to  $abcx$ . Hence  $cbxy$ , hence  $cbyx$ ; but  $cbyx \supset xybc \supset bcxy$ .

$abxy \supset yxba$  and  $bcxy \supset yxcb$ ;  $yxcb \cdot yxba \supset yxca$ ;  $yxca \supset acxy$ . So that we have  $abcx \cdot abcy \cdot abxy \cdot acxy \cdot bcxy$ . We have also  $axyz$  and  $bxyz$ , reading from left to right in  $C$ . It remains to be shown that  $cxyz$  holds.

$bcxy \supset cxyb$ ;  $cxyb \cdot z \supset cxyz$  or  $czyb$ ;  $czyb \supset bczy$ , contrary to  $bcyz$ . Hence  $cxyz$  if  $axyz$  and  $bxyz$ , and we have  $C(abc \dots xyz)$  for some permutation of the set of  $n+1$  elements, which is the theorem.

If  $Rabcd$  holds, then  $Rbcda$  and  $Rdcba$ . It follows that  $R$  holds for every tetrad which can be obtained by a cyclic permutation and its reverse. We may express, as previously, the holdure of  $R$  for these eight permutations by writing  $R'(abcd)$ . We may express by  $C'(abc \dots)$  the assertion that  $R'$  holds for any four elements in  $C'$  which occur in the order from left to right.

**THEOREM XVI.** Any  $n$  elements in  $K$  are such that for some permutation  $abc \dots$ ,  $C'(abc \dots)$  holds and such that  $R$  fails for every other permutation of any four of these elements.

$C(abc \dots)$  holds by XV. But  $Rabcd \supset R'(abcd)$ . Hence  $C'(abc \dots)$ . That all other ordered tetrads fail follows from Postulate 4.

We have then the following exhaustive characterization of any set of  $n$  elements in  $K$ .

(a) For some permutation  $C'(abc \dots)$  holds where  $a, b, \dots$  are all distinct.

(b) For every ordered tetrad of distinct elements which does not occur in  $C'$ ,  $R$  fails.

(c) For every ordered tetrad of the elements in  $C'$  which are not all distinct  $R$  fails.

(c) follows from postulate 00. If  $n$  is less than four (c) alone is relevant. It is clear that in view of (a), (b), and (c), any  $n$  elements in  $K$  have a determinate form; no further independent general laws can be added to the set 00-10.

All of the general laws 00-10 are universal in that they impose the same condition on every set of  $n$  elements in  $K$  and they are all hypothetical in that they are satisfied if  $K$  does not have at least  $n$  elements. We may,

however, have general laws which are not universal, though they must of course be hypothetical. Such laws are of the form "If there are at least  $n$  elements in  $K$ , then for some  $a, b, c, \dots, n, \phi(abc \dots n)$ ." And it is clear that a universal law implies the corresponding hypothetical particular, so that each member of the set 00-10 implies a corresponding particular function. Thus "For every distinct  $a, b, c, d$  in  $K, abcd \supset dcba$ " implies that if there are at least four elements in  $K$ , then for some  $a, b, c, d$  in  $K, abcd \supset dcba$ ; "For every distinct  $a, b, c, d, x$  in  $K, abcd \supset axcd$  or  $abcx$ " implies that if  $K$  has at least five elements, then for some  $a, b, c, d, x, abcd \supset axcd$  or  $abcx$ .\* Conversely, any property which can be assigned by a singly quantified first-order function and which belongs to at least one set of  $n$  elements in  $K$  belongs to every such set.

It is a characteristic of the set 00-10 that the holdure or failure of the relation  $R$  is left wholly undetermined for any tetrad  $abcd$  whose elements do not all belong to the class  $K$ . The foregoing theorems have been concerned solely with the determination of the form of any set of  $n$  elements all of which belong to  $K$ , and it is with respect to these elements that the set is complete as to general law. It seems to be commonly assumed, with regard to sets of this kind, that if a set of say four elements in the case of a tetradic  $R$  does not fall wholly within  $K$ , then  $R$  fails. It is sometimes suggested that  $R$  is undefined for such sets of elements. This can, however, mean no more than that  $R$  fails universally in such cases, for when we write "If  $a, b, c, d$  belong to  $K$ , then  $\phi(a, b, c, d)$ " it is presupposed that  $\phi(a, b, c, d)$  is significant even if  $a, b, c, d$  do not all belong to  $K$ . We may then subjoin to the set 00-10 the condition that if  $a, b, c, d$  do not all belong to  $K$ , then  $Rabcd$  fails. With this condition any set of  $n$  elements within the range of significance of the variables  $x, y$  in  $Rxy$  has a unique  $R$ -structure with respect to any general law.

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\* Cf. Huntington, Proceedings of the National Academy of Sciences, loc. cit., p. 688.

# INTEGERS AND BASIS OF A NUMBER FIELD\*

BY

N. R. WILSON

## I. INTRODUCTION

The existence theorems of the standard theory of algebraic numbers do not always lead to very practicable methods of computation in individual cases. Such methods of computation form the main subject of this paper. The basis set up appears at the same time to have some advantage of simplicity for theoretical purposes. The numbers of the field are first expressed in terms of a special set of integers, from which the basis is obtained in the third section. In the remaining sections, methods of computing this special set are discussed, illustrated by the general cubic.

Let  $A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0 = 0$  be the equation defining the field, all  $A$ 's being rational integers and  $A_n \neq 0$ . Let  $d$  be the greatest rational integer such that  $d^r / A_{n-r} A_n^{r-1} \dagger$  for all of  $r = 1, 2, \cdots, n$ . Then the substitution  $xd = zA_n$  reduces this equation to  $x^n + B_{n-1}x^{n-1} + \cdots + B_0 = 0$ , the  $B$ 's being rational integers. This form of the equation we call the *normal form*. Its defining features are (i) the coefficient of  $x^n$  is 1 and all  $B$ 's are rational integers; (ii) there exists no rational prime  $p$  such that  $p^r / B_{n-r}$  for all of  $r = 1, 2, \cdots, n$ . We may, if we please, make  $B_{n-1} = 0$ . These transformations, being rational, do not affect the field.

To minimize the verbiage, we use the following notation and terms, the latter mostly self-descriptive. The integers  $1, x, x^2, \cdots, x^{n-1}$  we call *ordinary integers*; also the sums and differences of such. These letters and  $y, z, Y, Z$  denote algebraic integers. The remaining letters,  $a, b, \cdots, w$ , and the corresponding capitals, denote rational integers,  $p$  being reserved for primes and  $p_1, p_2, \cdots$  denoting distinct primes. Greek letters denote rational numbers. If an algebraic integer is of the form  $\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ ,  $m \leq n-1$ ,  $\alpha_m \neq 0$ , and each  $\alpha$  in its lowest terms, we say that it is of degree  $m$  in  $x$ , abbreviated  $(\alpha_0, \alpha_1, \cdots, \alpha_m)$ . If also  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$  for each  $\alpha$  and  $\alpha_m = 1 \div D_m$  where  $D_m > 0$ , we call it a *reduced integer*. If  $y$  is a reduced integer and the denominators of the  $\alpha$ 's are powers of one and the same prime, we call  $y$  a *single-prime reduced integer*. If  $y$  is a single-prime reduced integer and  $\alpha_m = 1 \div p^t$  where  $t$  is the greatest possible, we say that  $y$  is a *maximal reduced integer* in  $p$  of degree  $m$ .

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† The symbol  $/$  throughout denotes "is a factor of."

**THEOREM I.** *If  $p^k$  is the highest power of  $p$  occurring in the denominator of any  $\alpha$  of the single-prime reduced integer  $\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ , then  $p^{2k}$  is a factor of the discriminant of the field equation.\**

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

Let  $y = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$  be the integer and  $y_2 = \alpha_0 + \alpha_1 x_2 + \cdots + \alpha_{n-1} x_2^{n-1}$ ,  $\cdots$ ,  $y_n = \alpha_0 + \alpha_1 x_n + \cdots + \alpha_{n-1} x_n^{n-1}$  be its conjugates. The determinant  $D$  on the left, when squared, gives the discriminant,  $\Delta$ . Let  $\alpha_r$ ,  $r \leq m$ , be any coefficient which contains  $p^k$  in its denominator. Multiplying the equations above by the co-factors of  $x^r$ ,  $x_2^r$ ,  $\cdots$ ,  $x_n^r$  in  $D$  and adding, we obtain on the right  $\alpha_r D$ , and on the left a determinant  $D'$  which is not affected, except for a change of sign, if we interchange a pair of conjugate roots. Hence  $D'^2$  also is a symmetric function of the roots; rational and, since the coefficient of  $x^n$  is 1, integral in the remaining coefficients  $B$ . Since  $\Delta = D^2$  and  $\Delta \neq 0$ ,  $\alpha_r^2 = D'^2 \div D^2$  or  $p^{2k}/\Delta$ .

**COROLLARY 1.** *If there exists a single-prime reduced integer in  $p$  of degree  $m$  in  $x$ , there exists a maximal reduced integer in  $p$  of degree  $m$  in  $x$ .*

For, since  $p^{2k}/\Delta$ ,  $k$  for  $\alpha_m$  is bounded and is a rational integer. Hence  $k$  must have a rational integral maximum,  $t$ .

**COROLLARY 2.** *The maximal reduced integers in a given field are finite in number.*

For  $m$  is restricted to the range  $0, 1, 2, \cdots, n-1$ ;  $p$  to the primes such that  $p^2/\Delta$ . For each  $m$  and  $p$  there can be only one maximum  $t$  of the last corollary. The coefficients  $\alpha$  are limited by the relation  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ . The number of each is finite, and therefore also the number of maximal reduced integers.

## II. EXPRESSION BY ORDINARY AND MAXIMAL REDUCED INTEGERS

We prove first that any integer can be expressed in terms of ordinary and maximal reduced integers, using not more than one of the latter for

\* The discriminant  $\Delta$  of the field equation throughout this paper denotes the product of the squared differences of the roots, without the additional numerical factor of some current definitions. We suppose always that  $\Delta \neq 0$ .

† For simplicity, when ambiguity is not likely to arise, "expressed in terms of" is used to abbreviate "expressed as a rational linear homogeneous function, with rational integral coefficients, of" except in enunciating theorems.

each  $m$  and  $p$ . If  $M = p_1^{k_1} p_2^{k_2} \cdots$  is the L. C. M. of the denominators of the coördinates in  $y = (\alpha_0, \alpha_1, \cdots, \alpha_m)$ ; and if  $u_1, u_2, \cdots$  are non-zero solutions of

$$M \left( \frac{u_1}{p_1^{k_1}} + \frac{u_2}{p_2^{k_2}} + \cdots \right) = 1$$

(the notation implying that they are rational integers); then  $y = u_1 y_1 + u_2 y_2 + \cdots$ , where

$$y_r = \frac{M}{p_r^{k_r}} y.$$

Each  $y_r$  contains in its denominators the same powers of  $p_r$  as in the corresponding denominators of  $y$ , and powers of  $p_r$  only. It will be sufficient therefore to prove the result for an integer  $y$ , all of whose denominators are powers of one prime  $p$ .

If  $y$  is such an integer of degree  $k$  in  $x$ , then  $y - (a_1, a_2, \cdots, a_k)$  is also an integer. Hence  $y$  is the sum of ordinary integers and an integer of the form

$$y' = \left( \frac{b_0}{p^{t_0}}, \frac{b_1}{p^{t_1}}, \cdots, \frac{b_m}{p^{t_m}} \right),$$

where  $-\frac{1}{2} < (b \div p^t) \leq \frac{1}{2}$ ,  $b_m \neq 0$  and prime to  $p$ . If  $ub_m + vp^{t_m} = 1$ ,  $uy' + vx^m$ , apart from ordinary integers, is a single-prime reduced integer,  $(\beta_0, \beta_1, \cdots, 1 \div p^{t_m})$ . Hence a maximal reduced integer in  $p$  of degree  $m$  in  $x$  exists (Theorem I, Corollary 1); viz.  $Y_m = (\gamma_0, \gamma_1, \cdots, 1 \div p^t)$ ,  $t \geq t_m$ .

Consider the integer  $y - b_m p^{t-t_m} Y_m$ . The coefficient of  $x^m$  is 0, so that it is of degree  $< m$  in  $x$ . If this difference is not expressible in terms of ordinary integers, we obtain in a similar manner a single-prime reduced integer,  $(\beta'_0, \beta'_1, \cdots, b'_s \div p^{u_s})$ , and a corresponding maximal reduced integer,  $Y_s = (\gamma_0, \gamma_1, \cdots, 1 \div p^u)$  of degree  $s$  in  $x$ ,  $s < m$  and  $u \geq u_m$ , and consider the difference  $(y - b_m p^{t-t_m} Y_m) - b_s p^{u-u_s} Y_s$ . Continuing this process so long as the difference is not expressible in terms of ordinary integers, we must, after  $m$  steps if not before, obtain a difference which is integral and of degree 0 in  $x$ ; i.e., a rational integer. Hence  $y$  is expressible in terms of  $Y_m, Y_s, \cdots$  and ordinary integers. As all maximal reduced integers in  $p$  of degree  $m$  in  $x$  have the same highest coefficient, any one with the same  $m$  and  $p$  may be used in making these reductions.

**THEOREM IIa.** *All integers of the field can be expressed as rational linear homogeneous functions with rational integral coefficients of ordinary integers and maximal reduced integers, the latter consisting of one selected arbitrarily from those in each prime and for each degree in  $x$  for which such exist.*

COROLLARY. *The difference between two integers in  $p$ ,*

$$\left( \frac{a_0}{p^{t_0}}, \frac{a_1}{p^{t_1}}, \dots, \frac{a_m}{p^{t_m}} \right)$$

and

$$\left( \frac{b_0}{p^{u_0}}, \frac{b_1}{p^{u_1}}, \dots, \frac{b_m}{p^{u_m}} \right),$$

of the same degree in  $x$  and with the same highest coefficient, is expressible in terms of ordinary integers and maximal reduced integers in  $p$ , both of lower degree in  $x$ .

THEOREM IIb. *If  $p$  is a prime occurring in the denominator of some co-ordinate of an integer, then (i) there exists exactly one maximal reduced integer  $Y_r$ , in  $p$  of lowest degree  $r$  in  $x$ ,  $r > 0$ ; (ii) if  $r < n - 1$ , there exist maximal reduced integers in  $p$  of degrees  $r + 1, r + 2, \dots, n - 1$  in  $x$ ; and (iii) for each  $u$ ,  $0 < u < t$ , there is one and but one single-prime reduced integer,  $(\beta_0, \beta_1, \dots, 1 \div p^u)$  of degree  $r$  in  $x$  differing from  $p^{t-u}Y$  by ordinary integers, where  $Y_r = (\gamma_0, \gamma_1, \dots, 1 \div p^t)$ .*

As in Theorem Ia, there exists a single-prime reduced integer and therefore a maximal reduced integer in  $p$  of some degree  $m$  in  $x$ . Since  $m$  has 0 for a lower bound and is a rational integer, there exists at least one of some lowest degree  $r$ . We must have  $r > 0$ ; for any rational number which is also an integer must be a rational integer. If there were two of this lowest degree in  $x$ , since their highest coefficients are the same, their difference leads to a single-prime reduced integer of lower degree, since  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ . From Corollary 1, Theorem Ia, we should then have a maximal reduced integer of degree  $r'$ ,  $r' < r$ , the least degree in  $x$ .

If  $Y_r$  is this one and  $r < n - 1$ , then  $xY_r, x^2Y_r, \dots, x^{n-r+1}Y_r$  are single-prime reduced integers in  $p$  of degrees  $r + 1, r + 2, \dots, n - 1$ , in  $x$ , hence, by the same corollary, there are maximal reduced integers of these degrees in  $x$ . The integer given in (iii) shows that there is at least one integer for each  $u$ ; that there cannot be more than one follows exactly as in (i).

COROLLARY. *The maximal reduced integers of degree  $r + 1$  in  $x$  are of the form  $Y_{r+1} + mY_r$ , where  $m = 0, 1, 2, \dots, (p^t - 1)$  and  $Y_{r+1}$  is any one of them; those of degree  $r + 2$  in  $x$  of the form  $Y_{r+2} + nY_{r+1} + mY_r$ , where also  $n = 0, 1, 2, \dots, (p^u - 1)$ ,  $1 \div p^u$  being the highest coefficient in  $Y_{r+2}$ , any one of them, and so on.*

THEOREM IIc. *If  $(a_0 \div p^{t_0}, a_1 \div p^{t_1}, \dots, 1 \div p^{t_m})$  is a maximal reduced integer in  $p$  of degree  $m$  in  $x$  then  $t_s \leq t_m$  for  $s < m$ .*

For the lowest degree  $r$  of the preceding theorem, let

$$Y_r = \left( \frac{a_0}{p^{t_0}}, \frac{a_1}{p^{t_1}}, \dots, \frac{1}{p^{t_r}} \right).$$

If possible, let any  $t_s, s < r$ , be the last index exceeding  $t_r$ . Then the integer  $p^{t_r} Y_r$  leads to the single-prime reduced integer  $(\beta_0, \beta_1, \dots, 1 \div p^u)$ , of degree  $s$  in  $x$ , where  $u = t_s - t_r$ . There is therefore a maximal reduced integer of degree  $s$  in  $x, s < r$ , contrary to hypothesis with respect to  $r$ .

For degree  $r + 1$  in  $x$  if  $r < n - 1$ , we have

$$xY_r = \frac{a_0}{p^{t_0}} x + \frac{a_1}{p^{t_1}} x^2 + \dots + \frac{1}{p^{t_r}} x^{r+1}.$$

If

$$Y_{r+1} = \frac{b_0}{p^{u_0}} + \frac{b_1}{p^{u_1}} x + \dots + \frac{b_r}{p^{u_r}} x^r + \frac{1}{p^{t_{r+v}}} x^{r+1}$$

is a maximal reduced integer of degree  $r + 1$  in  $x$ , the difference  $p^v Y_{r+1} - xY_r$  is of degree  $< r + 1$ , and therefore of the form  $cY_r$ . Examining the coefficient of  $x^k$ , the degree of  $p$  in  $(b_k \div p^{u_k-v}) - (a_{k-1} \div p^{t_{k-1}})$  cannot exceed  $t_r$ . Since  $t_{k-1} \leq t_r$ , we have  $u_k - v \leq t_r$  or  $u_k < t_r + a$ , the index of the highest coefficient in  $Y_{r+1}$ . Replacing  $r$  by  $r + 1$ , the same result follows in similar fashion for  $Y_{r+2}$ , the sole change being that the difference is of the form  $cY_{r+1} + dY_r$  instead of  $cY_r$ . Similarly for all degrees in  $x$  up to  $n - 1$ . (This property obviously does not hold for single-prime reduced integers in general unless all maximal reduced integers of degrees  $< m$  in  $x$  are of degree 1 in  $p$ .)

**COROLLARY.** If  $t_r, t_{r+1}, \dots, t_{n-1}$  denote the degrees of  $p$  in the highest coefficients for a series of maximal reduced integers in  $p, Y_r, Y_{r+1}, \dots, Y_{n-1}$ , of the degrees in  $x$  indicated by the subscripts, then  $t_r \leq t_{r+1} \leq \dots \leq t_{n-1}$ ; if  $(n-1) \div r \geq 2$ , then  $kt_r \leq t_m$  for  $m \geq kr$ , where  $k$  is any integer  $\geq (n-1) \div r$ .

For the degree of  $p$  in the highest coefficient of  $Y_{m+1}$  must be at least the degree in  $xY_m$  and therefore not less than the degree in  $Y_m$ . Also if  $m \geq kr$ , the degree of  $p$  in the highest coefficient of  $Y_m$  must be at least that in  $Y_r^k$ .

**THEOREM II d.** If  $r, r < n - 1$ , be the lowest degree in  $x$  for which a maximal reduced integer in  $p$  exists, then there exist single-prime reduced integers of every degree  $m$  in  $x, r < m \leq n - 1$ , such that  $\alpha_r = \alpha_{r+1} = \dots = \alpha_{m-1} = 0$ .

For, let

$$Y_m = \left( \alpha_0, \dots, \frac{b}{p^{t_{m-1}}}, \frac{1}{p^{t_m}} \right) \text{ and } Y_{m-1} = \left( \alpha_0, \dots, \frac{1}{p^{u_{m-1}}} \right)$$

be maximal reduced integers of degrees  $m$  and  $m-1$  in  $x$ . Then  $p^v Y_m - b p^w Y_{m-1}$ , where  $v = t_m - t_r$  and  $w = t_m - t_{m-1} + u_{m-1} - t_r$ , has  $\alpha_{m-1} = 0$  and is an integer, since by the preceding theorem and corollary the indices  $v$  and  $w$  are not negative. Similarly, by subtracting proper multiples of  $Y_{m-2}, \dots, Y_r$ , we may reduce the coefficients  $\alpha_{m-2}, \dots, \alpha_r$  to 0. (The coefficients can obviously be given any arbitrary values, those obtained being most convenient for use with § IV.)

### III. CONSTRUCTING A BASIS

Since the number of different primes which can occur in the denominator of an integer is finite (Theorem I), and for each there is a maximal reduced integer of lowest degree  $r'$  in  $x$ ,  $r' > 0$ , there will be a lowest degree  $r$ ,  $r > 0$ , for which any such occurs. For any  $k$ ,  $r \leq k \leq n-1$ , let

$$y_{k,1} = \left( \alpha_0, \dots, \frac{1}{p_1^{t_1}} \right), \quad y_{k,2} = \left( \alpha_0, \dots, \frac{1}{p_2^{t_2}} \right), \dots$$

be any selection of maximal reduced integers of degree  $k$  in  $x$ , one for each distinct prime for which such occur. Let  $P_k = p_1^{t_1} p_2^{t_2} \dots$  and let  $v_1, v_2, \dots$  be non-zero solutions of

$$P_k \left( \frac{v_1}{p_1^{t_1}} + \frac{v_2}{p_2^{t_2}} + \dots \right) = 1.$$

Let  $Z_k$  be the reduced integer derived from  $v_1 y_{k,1} + v_2 y_{k,2} + \dots$  by removing ordinary integers as in §II. Then  $Z_k$  is of the form  $(\beta_0, \beta_1, \dots, 1 \div P_k)$ . The coefficient of  $x^k$  in  $P_k Z_k$  is 1, and, by Theorem IIc, lower powers of  $x$  reduce to ordinary integers. Hence  $x^k$  is expressible in terms of  $Z_k$  and ordinary integers of degrees  $< k$ . Also  $(P_k \div p_s^{t_s}) Z_k$  is of the form  $(\beta_0', \dots, 1 \div p_s^{t_s})$ . The corresponding reduced integer,

$$Y_{k,s} = \left( \beta_0'', \dots, \frac{1}{p_s^{t_s}} \right),$$

is therefore a maximal reduced integer in  $p_s$ . Hence  $Y_{k,s}$  is expressible in terms of  $Z_k$  and ordinary integers of degrees  $< k$  in  $x$ .

For the basis,  $\omega_1, \omega_2, \dots, \omega_r, \omega_{r+1}, \dots, \omega_n$ , we take  $1, x, \dots, x^{r-1}, Z_r, Z_{r+1}, \dots, Z_{n-1}$ . These are linearly independent since each contains a power of  $x$  higher than the preceding. By Theorem IIa, every integer can

be expressed in terms of ordinary integers and the maximal reduced integers,  $Y_{k,1}, Y_{k,2}, \dots$  for  $r \leq k \leq n-1$ . Ordinary integers of degrees  $< r$  in  $x$  are exactly  $\omega_1, \omega_2, \dots, \omega_r$ . We have proved in the preceding paragraph that  $x^r$  and  $Y_{r,1}, Y_{r,2}, \dots$  can be expressed in terms of  $Z_r$  and ordinary integers of degrees  $< r$  in  $x$ ; i.e. in terms of  $\omega_{r+1}$  and  $\omega_r, \dots, \omega_1$ . Also that  $x^{r+1}$  and  $Y_{r+1,1}, Y_{r+2,1}, \dots$  can be expressed in terms of  $Z_{r+1}$  and ordinary integers of degree  $< r+1$  in  $x$ ; i.e. in terms of  $\omega_{r+2}$  and  $\omega_{r+1}, \omega_r, \dots, \omega_1$ . And so on. Hence the given set actually form a basis. An integer of degree  $m$  in  $x$  can in fact be expressed in terms of  $\omega_1, \dots, \omega_{m+1}$ .

The usual theorems follow at once. If  $\omega'_i = \sum_{j=1}^n c_{ij} \omega_j$ ,  $1 \leq i \leq n$ , are linearly independent, the  $c$ 's being rational integers, we can express each  $\omega_i$  rationally and integrally in terms of the set  $\omega'$ , provided  $|c_{ij}|^2 = 1$ . Hence with this condition the set  $\omega'$  also form a basis. The discriminant  $\Delta_1(\omega'_1, \dots, \omega'_n) = |c_{ij}|^2 \Delta_1(\omega_1, \dots, \omega_n)$ , and since  $|c_{ij}|$  is rational, integral and  $\neq 0$ ,  $|c_{ij}|^2 \geq 1$ . Hence  $\Delta_1(\omega_1, \dots, \omega_n)$  is a minimum, and, if any  $\Delta_1(\omega'_1, \dots, \omega'_n) = \Delta_1(\omega_1, \dots, \omega_n)$ ,  $|c_{ij}|^2 = 1$  and the set  $\omega'$  form a basis. Finally, since we have seen that the minimum degree for which maximal reduced integers exist is greater than 0, 1 is always a member of a basis constructed as above.

We may deduce a relation between the discriminant of the field and of the defining equation. In the field discriminant below,  $\omega_1^{(1)} = 1, \omega_2^{(1)} = x, \dots, \omega_r^{(1)} = x^{r-1}, \omega_{r+1}^{(1)} = \alpha_0 + \alpha_1 x + \dots + (1 \div P_r)x^r, \dots, \omega_n^{(1)} = \delta_0 + \delta_1 x + \dots + (1 \div P_{n-1})x^{n-1}$ . The columns,  $\omega_2^{(i)}, \dots, \omega_n^{(i)}, 1 < i \leq n$ , are obtained by replacing  $x$  by its conjugates  $x_i$ . Keeping the first row unaltered, by subtracting proper multiples of preceding rows, we

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \omega_2^{(1)} & \omega_2^{(2)} & \dots & \omega_2^{(n)} \\ \omega_3^{(1)} & \omega_3^{(2)} & \dots & \omega_3^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^{(1)} & \omega_n^{(2)} & \dots & \omega_n^{(n)} \end{vmatrix}^2 = \Delta_1$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x^{r-1} & x_2^{r-1} & \dots & x_n^{r-1} \\ \frac{1}{p^r} x^r & \frac{1}{p^r} x_2^r & \dots & \frac{1}{p^r} x_n^r \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_{n-1}} x^{n-1} & \frac{1}{p_{n-1}} x_2^{n-1} & \dots & \frac{1}{p_{n-1}} x_n^{n-1} \end{vmatrix}^2 = \Delta_1$$

denotes the discriminant of the equation defining the field.

obtain the second determinant, on the left. (Thus, for the  $s$ th row,  $s > r$ , on subtracting the proper multiple of the  $(s-1)$ th row, we eliminate the terms  $x_i^{s-1}$ ; from the difference, the proper multiple of the  $(s-2)$ th row, we eliminate  $x_i^{s-2}$ ; and so on.) The latter determinant on inspection is seen to be  $\Delta \div P_r^2 P_{r+1}^2 \dots P_{n-1}^2$ , where  $\Delta$

THEOREM III. If  $\Delta_1, \Delta$  denote the discriminants of the field and of the equation defining the field, and  $P_r, P_{r+1}, \dots, P_{n-1}$  are the denominators of the highest coefficients of the elements of the basis as above determined, other than those which are ordinary integers, then  $\Delta = P_r^2 P_{r+1}^2 \dots P_{n-1}^2 \Delta_1$ .

Since  $\Delta_1$  must be a rational integer,

COROLLARY. If  $P_r, P_{r+1}, \dots, P_{n-1}$  are the denominators of the highest coefficients of elements of the basis as above determined, other than those which are ordinary integers,  $(P_r P_{r+1} \dots P_{n-1})^2$  is a factor of  $\Delta$ .

#### IV. RELATIONS AMONG THE COÖRDINATES OF AN INTEGER; FIRST METHOD

If  $y = \alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1}$  is an integer, consider the product-equation

$$\Pi \{y - (\alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1})\} = 0,$$

where  $x$  runs through the complete set of  $n$  conjugates, the field being given by the equation  $x^n - B_{n-1} x^{n-1} + \dots + B_0 = 0$ . The coefficients of this equation, arranged in powers of  $y$ , are symmetric functions of the  $x$ 's. Since the coefficient of  $x^n$  is 1, they are rational integral functions of the  $B$ 's. Since the  $\alpha$ 's are rational, they are rational numbers. They are also integers, being the sums and products of integers. Hence they are rational integers.

The absolute term of this equation is the eliminant of  $\alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1}$  and  $x^n + B_{n-1} x^{n-1} + \dots + B_0$ , obtained by symmetric functions. Since the coefficients of  $\alpha_0^n$  in the eliminant thus obtained and in the eliminant obtained in the more convenient Sylvester form below are the same, the eliminants are identical. We denote this eliminant by  $E(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ . The left side of the above equation in  $y$  is  $E(\alpha_0 - y, \alpha_1, \dots, \alpha_{n-1})$ .

THEOREM IVa. If  $y = \alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1}$  is an integer, then all of

$$\frac{1}{m!} \frac{\partial^m E}{\partial \alpha_0^m} \quad (m = 0, 1, 2, \dots, n-1)$$

and all of

$$\frac{1}{2!(n-2)!} \frac{\partial^{n-1} E}{\partial \alpha_0^{n-2} \partial \alpha_r} \quad (r = 1, 2, \dots, n-1),$$

are rational integers; conversely, if all of

$$\frac{1}{m!} \frac{\partial^m E}{\partial \alpha_0^m} \quad (m = 0, 1, 2, \dots, n-1)$$

are rational integers, then  $y$  is an integer.

$$E = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ B_0 & B_1 & B_2 & B_3 & 1 & 0 & 0 \\ 0 & B_0 & B_1 & B_2 & B_3 & 1 & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & B_3 & 1 \end{vmatrix} \quad \begin{array}{l} \text{On inspection of the sample} \\ \text{determinant on the left, we have} \\ \text{that the coefficient of } \alpha_0^n \text{ is 1 and all} \\ \text{partial differential coefficients with} \\ \text{respect to the } \alpha\text{'s of order higher} \\ \text{than } n \text{ vanish. Also} \\ \frac{1}{2!(n-2)!} \frac{\partial^n E}{\partial \alpha_0^{n-2} \partial \alpha_r^2} (1 \leq r \leq n-1) \end{array}$$

is a rational integer. For the terms of  $E$  which do not vanish from the differentiation are those involving  $\alpha_0^{n-2} \alpha_r^2$ . The factors arising from the indices on differentiating such terms will cancel  $2!(n-2)!$ . The remaining factors are minors from the lowest  $(n-1)$  rows and therefore rational integers.

The equation for  $y$  is  $E(\alpha_0 - y, \alpha_1, \dots, \alpha_{n-1}) = 0$ , or

$$y^n - \frac{1}{(n-1)!} \frac{\partial^{n-1} E}{\partial \alpha_0^{n-1}} y^{n-1} + \frac{1}{(n-2)!} \frac{\partial^{n-2} E}{\partial \alpha_0^{n-2}} y^{n-2} - \dots \pm \frac{\partial E}{\partial \alpha_0} y \mp E = 0,$$

on expanding by Taylor's Theorem. If the coefficients of the powers of  $y$  are rational integers,  $y$  is an integer, proving the converse part of the theorem. For the reasons stated in the first paragraph, these coefficients are rational integers if  $y$  is an integer, proving the first part of the necessary conditions. Hence in particular,

$$\frac{1}{(n-2)!} \frac{\partial^{n-2} E}{\partial \alpha_0^{n-2}}$$

is a rational integer. But if  $(\alpha_0, \alpha_1, \dots, \alpha_r, \dots, \alpha_{n-1})$  is an integer, so also is  $(\alpha_0, \alpha_1, \dots, \alpha_r + 1, \dots, \alpha_{n-1})$ . Hence we have the rational integer

$$\begin{aligned} & \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial \alpha_0^{n-2}} E(\alpha_0, \alpha_1, \dots, \alpha_r + 1, \dots, \alpha_{n-1}) \\ &= \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial \alpha_0^{n-2}} E(\alpha_0, \alpha_1, \dots, \alpha_r, \dots, \alpha_{n-1}) + \frac{1}{(n-2)!} \frac{\partial^{n-1} E}{\partial \alpha_0^{n-2} \partial \alpha_r} \\ & \quad + \frac{1}{2!(n-2)!} \frac{\partial^{n-2} E}{\partial \alpha_0^{n-2} \partial \alpha_r^2}. \end{aligned}$$

Since the first and last expressions on the right are rational integers, so also is the middle expression.

In applying this result, it will be found simplest to obtain  $E$  by forming  $yx, yx^2, \dots, yx^{n-1}$ , reducing to degree  $(n-1)$  by the field equation.\* Thus for the cubic field,  $x^3 + Qx + R = 0$ , we have  $y = \alpha_0 + \alpha_1x + \alpha_2x^2$ ,  $yx = -R\alpha_2 + (\alpha_0 - \alpha_2Q)x + \alpha_1x^2$ ,  $yx^2 = -R\alpha_1 - (\alpha_1Q + \alpha_2R)x + (\alpha_0 - \alpha_2Q)x^2$ , and obtain the rational integers from Theorem IVa:  $\alpha_0^3 - 2\alpha_0^2\alpha_2Q + \alpha_0\alpha_1^2Q + 3\alpha_0\alpha_1\alpha_2R + \alpha_0\alpha_2^2Q^2 - \alpha_1^3R - \alpha_1\alpha_2^2QR + \alpha_2^3R^2$ ;  $3\alpha_0^2 - 4\alpha_0\alpha_2Q + \alpha_1^2Q + 3\alpha_1\alpha_2R + \alpha_2^2Q^2$ ;  $3\alpha_0 - 2\alpha_2Q$ ;  $2\alpha_1Q + 3\alpha_2R$ ;  $\alpha_0Q - 3\alpha_1R$ ; all but the last are obtained by direct differentiation, the last being simplified by the second preceding.

COROLLARY. *The Taylor expansion of*

$$\frac{1}{m!} \frac{\partial^m}{\partial \alpha_0^m} E(\alpha_0, \dots, \alpha_{r-1}, \alpha_r + h, \alpha_{r+1}, \dots, \alpha_{n-1})$$

*in powers of  $h$ , less the first and last terms, is a rational integer for all rational integral values of  $h$ .*

The rational integers obtained by the corollary can be simplified by the linear equations obtained from the theorem as in the last case above for the cubic, or by the following theorem.

THEOREM IVb. *If  $A_1h + A_2h^2 + \dots + A_mh^m$  is a rational integer for  $m$  unequal rational integral non-zero values of  $h$ , then every  $A_k$  is rational, its denominator being a factor of the product of the values of  $h$  and their differences; in particular, if this expression is a rational integer for  $h = 1, 2, \dots, m$ , these denominators are factors of  $m(m-1)^2(m-2)^3 \dots 2^{n-1}$ .*

Substituting the given values,  $h_1, \dots, h_m$ , in  $A_1h + A_2h^2 + \dots + A_mh^m$ , and solving for the  $A$ 's, the numerators are rational integers and the denominator is  $D$  on the left. By inspection,  $h_1, h_2, \dots, h_m$  and all differences are seen to be factors of  $D$ , accounting for all literal factors. From the principal diagonal, the remaining numerical factor is  $\pm 1$ . Hence the denominators of the  $A$ 's are factors of their product. In the particular case, these give the product stated. Applied to  $E$  for the cubic  $x^3 + Qx + R = 0$ , we obtain the rational integer

$$h \frac{\partial E}{\partial \alpha_1} + \frac{h^2}{2!} \frac{\partial^2 E}{\partial \alpha_1^2},$$

\* This is equivalent to expanding the eliminant given above by Chio's pivotal method, applied to the 1's in the principal diagonal.

so that, by Theorem IV b,  $\partial E/\partial \alpha_1$  is a rational integer or one half such. As we are concerned only with maximal reduced integers, we have that, unless  $p = 2$ ,

$$\frac{\partial E}{\partial \alpha_1} = 2\alpha_0\alpha_1Q + 3\alpha_0\alpha_2R - 3\alpha_1^2R - \alpha_2^2QR$$

for such is a rational integer; if  $p = 2$ , it is a rational integer divided by 2.

We have seen that there is no maximal reduced integer of degree 0 in  $x$  (§III). If

$$Y_1 = \frac{a_0}{p^{t_0}} + \frac{1}{p^{t_1}}x$$

is a maximal reduced integer of degree 1 in  $x$ , then  $Y_2$  is of degree at least  $2t_1$  in  $p$ ,  $Y_3$  at least  $3t_1, \dots, Y_{n-1}$  at least  $(n-1)t_1$  (Corollary, Theorem IIc). Hence  $p^{n(n-1)t_1}/\Delta$  (Theorem III). By Theorem IVa,

$$\frac{1}{(n-1)!} \frac{\partial^{n-1} E}{\partial \alpha_0^{n-1}} \quad \text{and} \quad \frac{1}{2!(n-2)!} \frac{\partial^{n-1} E}{\partial \alpha_0^{n-2} \partial \alpha_1}$$

are rational integers. If  $B_{n-1} = 0$ , from  $E$  as given at the beginning of this section these reduce to  $n\alpha_0$  and  $B_{n-2}\alpha_1$  respectively. Hence

**THEOREM IVc.** *For the field  $x^n + B_{n-2}x^{n-2} + \dots + B_0 = 0$ , the maximal reduced integer  $(a_0 \div p^{t_0}, a_1 \div p^{t_1})$  can exist only if (i)  $p^{n(n-1)t_1}/\Delta$ , (ii)  $p^{t_1}/B_{n-2}$  and (iii)  $a_0 = 0$  or  $p^{t_0}/n$ .*

**COROLLARY.** *If  $Y = (\alpha_0, \dots, 1 \div p^{t_m})$  of degree  $m$  in  $x$  is a maximal reduced integer and  $n-1 = qm + R$ ,  $0 \leq R < m$ , then  $p^v/\Delta$ , where  $v = \{mq(q-1) + 2q(r+1)\}t_m$ .*

For  $Y_m, Y_{m+1}, \dots, Y_{2m-1}$  are of degrees at least  $t_m$  in  $p$ ;  $Y_{2m}, Y_{2m+1}, \dots, Y_{3m-1}$ , at least  $2t_m, \dots$ ;  $Y_{qm}, \dots$  at least  $qt_m$ .

#### V. RELATIONS AMONG THE COORDINATES OF AN INTEGER; SECOND METHOD

The formulas of §IV have been determined by observing that, if  $y$  is an integer, so also is  $y + x^r$ . These will be found sufficient to determine the set of maximal reduced integers in numerical cases and thence a basis,

possible primes in the denominators of the former being determined by Theorem I. We may also obtain useful relations by observing that, if  $y$  is an integer, so also is  $yx$ .

**THEOREM V.** *When a maximal reduced integer in  $p$  of degree  $n - 1$  in  $x$  exists but none in  $p$  of lower degree in  $x$ , there exists a single-prime reduced integer,*

$$y = \frac{1}{p} (a_0, a_1, \dots, a_{n-2}, 1),$$

*such that either (i)  $yx$  reduces to ordinary integers and  $a_{n-2} \equiv B_{n-1}$ ,  $a_{n-3} \equiv B_{n-2}$ ,  $\dots$ ,  $a_0 \equiv B_1$ ,  $0 \equiv B_0$ , all (mod  $p$ ), or (ii)  $yx$  does not reduce to ordinary integers and  $a_{n-2} \not\equiv B_{n-1}$ ,  $a_{n-2}(a_{n-2} - B_{n-1}) \equiv a_{n-3} - B_{n-2}$ ,  $\dots$ ,  $a_{r+1}(a_{n-2} - B_{n-1}) \equiv a_r - B_{r+1}$ ,  $\dots$ ,  $a_1(a_{n-2} - B_{n-1}) \equiv a_0 - B_1$ ,  $a_0(a_{n-2} - B_{n-1}) \equiv -B_0$ , all (mod  $p$ ).*

For on reducing  $x^n$  in  $yx$  by the field equation, we have

$$yx = -\frac{B_0}{p} + \frac{a_0 - B_1}{p}x + \dots + \frac{a_r - B_{r+1}}{p}x + \dots + \frac{a_{n-3} - B_{n-2}}{p}x^{n-2} + \frac{a_{n-2} - B_{n-1}}{p}x^{n-1}.$$

If  $a_{n-2} \equiv B_{n-1} \pmod{p}$ , since there is no single-prime reduced integer in  $p$  of degree  $< n - 1$  in  $x$ ,  $yx$  reduces to ordinary integers and the remaining coefficients must be rational integers, leading to the congruences in (i). If  $a_{n-2} \not\equiv B_{n-1}$ ,  $yx$  cannot, but  $yx - (a_{n-2} - B_{n-1})y$  must reduce to ordinary integers. The coefficients of each power of  $y$  in this difference must be rational integers, leading to the congruences in (ii).

**COROLLARY 1.** *If  $B_0 \not\equiv 0 \pmod{p}$  and a maximal reduced integer in  $p$  of degree  $n - 1$  in  $x$  exists but none in  $p$  of lower degree in  $x$  then, if any  $a_r \equiv B_{r+1} \pmod{p}$ ,  $a_{r+1} = 0$ .*

For, in this case, by the last congruence in (i) above,  $yx$  cannot reduce to ordinary integers; whence the result follows from (ii).

**COROLLARY 2.** *If  $p$  is a factor of each of  $B_0, B_1, \dots, B_{n-1}$ , and a maximal reduced integer in  $p$  of degree  $n - 1$  in  $x$  exists but none in  $p$  of lower degree in  $x$ , then  $x^{n-1} \div p$  is an integer.*

If so,  $yx$  reduces to ordinary integers. For, if not,  $a_{n-2} \not\equiv B_{n-1} \pmod{p}$ .

From the last congruence of (ii), remembering that  $-\frac{1}{2} < (a_0 \div p) \leq \frac{1}{2}$ , we have that  $a_0 = 0$  since  $p/B_0$ . From the second last,  $a_1 = 0$  and so on to  $a_{n-2} = 0$ , contradicting  $a_{n-2} \not\equiv B_{n-1} \pmod{p}$ . If  $yx$  reduces to ordinary integers,  $a_{n-2} \equiv B_{n-1} \pmod{p}$ , or  $a_{n-2} = 0$  since  $p/B_{n-1}$ . From the congruences of (i), we have  $a_{n-3} = 0, \dots, a_0 = 0$ , whence the integer must be  $x^{n-1} \div p$ .

**COROLLARY 3.** *If  $p$  is a factor of  $B_0, B_1, \dots, B_r$ , but not of  $B_{r+1}$ , and there is a maximal reduced integer in  $p$  of degree  $n-1$  but none in  $p$  of lower degree in  $x$ , then  $a_0 = a_1 = \dots = a_{r-1} = 0$  and either (i)  $a_r \neq 0$  if  $a_{n-2} \equiv B_{n-1} \pmod{p}$  or (ii)  $a_r \equiv 0, a_{r+1} \neq 0$  if  $a_{n-2} \not\equiv B_{n-1} \pmod{p}$ .*

The proof follows the same lines as in Corollary 2.

**COROLLARY 4.** *If  $p$  is a factor of  $B_{n-1}$  but not of  $B_0$ , then  $a_0 \neq 0$  and  $a_{n-2} \neq 0$ .*

For  $yx$  can not reduce to ordinary integers. Hence  $a_{n-2} \not\equiv B_{n-1}$ , or  $a_{n-2} \neq 0$ . Since  $a_0(a_{n-2} - B_{n-1}) \equiv -B_0 \pmod{p}$ ,  $a_0 \neq 0$ .

A two-fold use may be made of this theorem and its corollaries. First, many cases may be excluded on inspection before applying Theorem IVa. Second, when a maximal reduced integer in  $p$  exists, the single-prime reduced integer obtained serves as a starting point in building up the former. The theorem can be enunciated so as to cover cases in which the powers of  $p$  in the denominators are higher than the first, but, in practice, it will be found more convenient to obtain the integer of the theorem and to apply the method to obtain integers with higher powers, using Theorem IIb. Using the corollary to Theorem IIa, it may also be enunciated to cover cases in which there are maximal reduced integers of degree  $< n-1$  in  $x$ . The corollary to Theorem IIc, however, furnishes a single-prime reduced integer at once, from which the maximal reduced integer can be built up. Both these methods are illustrated in the next section, the former by the case  $p = 2$ , etc., and the latter by the case  $p = 3$ .

## VI. APPLICATION TO THE CUBIC FIELD

In the first instance we suppose that the cubic is reduced to  $x^3 + Qx + R = 0$  in the normal form of §I. If  $y = (\alpha_0, \alpha_1, \alpha_2)$  is an integer, from Theorem IVa, we have the rational integers

$$(Ia) \ 3\alpha_0 - 2\alpha_2Q; \quad (Ib) \ 2\alpha_1R + 3\alpha_2R; \quad (Ic) \ \alpha_0Q - 3\alpha_1R;$$

$$(II) \ 3\alpha_0^2 - 4\alpha_0\alpha_2Q + \alpha_1^2Q + 3\alpha_1\alpha_2R + \alpha_2^2Q^2;$$

$$(III) \ \alpha_0^3 - 2\alpha_0^2\alpha_2Q + \alpha_0\alpha_1^2Q + 3\alpha_0\alpha_1\alpha_2R + \alpha_0\alpha_2^2Q^2 - \alpha_1^3R - \alpha_1\alpha_2^2QR + \alpha_2^3R^2.$$

That (Ia), (II) and (III) are rational integers is a condition sufficient to make  $y$  an integer. We use the standard notation,  $(a_0 \div p^t, a_1 \div p^m, a_2 \div p^n)$  for single-prime reduced integers. The discriminant,  $\Delta$ , equals  $-4Q^3 - 27R^2$ .

If there is a maximal reduced integer,  $y = (a_0 \div p^t, a_1 \div p^m)$ , of degree 1 in  $x$ , it cannot have  $a_0 = 0$ . For, by (II) above,  $p^{2m}/Q$ , and by (III)  $p^{3m}/R$ ; which cannot occur if the equation is in the normal form. Hence further  $m \geq t$ ; for, if so, the reduced integer derived from  $p^{m-t}y$  has  $a_0 = 0$ . By Theorem IVc, since  $a_0 \neq 0$ ,  $p^t/3$  and  $p^m/Q$ . Hence  $p = 3$ ,  $t = 1 = m$ , and  $3/Q$ . Since  $-\frac{1}{2} < (a_0 \div 3) \leq \frac{1}{2}$ ,  $a_0 = \pm 1$ . Substituting these values, (Ia) is satisfied, (II) is satisfied only if  $Q \equiv -3 \pmod{9}$ , and (III) only if  $R \equiv \pm(Q+1) \pmod{27}$  according as  $a_0 = \pm 1$ . We have therefore the maximal reduced integer  $\frac{1}{3}(x \pm 1)$ .

We dispose first of the cases in which there is a maximal reduced integer of degree 2 in  $x$  but none of degree 1. If both  $p/Q$  and  $p/R$ , by Corollary 2, Theorem V,  $x^2 \div p$  is an integer. Conditions (Ia), (II) and (III) are satisfied if  $p^2/R$ . If  $x^2 \div p$  is not maximal, by Theorem IIb, we must have an integer of the form  $(a_0 \div p, a_1 \div p, 1 \div p^2)$ . Since the equation is in its normal form, we cannot have  $p^2/Q$  and  $p^3/R$ . But, since  $p^1/\Delta = -4Q^3 - 27R^2$  (Theorem I),  $p^2/Q$  unless  $p = 2$ . Hence, from (Ia), unless  $p = 2, 3$ , we have  $a_0 = 0$  and, from (II) if  $a_1 \neq 0$ , or (III) if  $a_1 = 0$ ,  $p^3/R$ . If  $p = 3$ , we have from (Ia) that  $3^2/Q$  and we proved above that  $3^2/R$ . From (III),  $3/\alpha_0^3$ , or  $a_0 = 0$ . Hence, again from (III),  $3^3/R^2$ , so that  $3^3/R$ . Finally, if  $p = 2$ , from (Ia)  $a_0 = 0$  and from (II), written as a congruence,  $6a_1R + Q^2 \equiv 0 \pmod{16}$ . Since  $2^2/R$ ,  $2^3/Q^2$  or  $2^2/Q$ ; also, unless  $2^3/R$ ,  $a_1 = 0$ . If so, from (III),  $2^3/R$ . Hence there can be no integer of the form  $(a_0 \div p, a_1 \div p, 1 \div p^2)$ .

If neither  $p/Q$  nor  $p/R$ , in the integer referred to in Theorem V,  $a_0 \neq 0$ ,  $a_1 \neq 0$  (Corollary 4). Hence any maximal reduced integer is homogeneous in  $p$ . Since  $p^2/\Delta = -4Q^3 - 27R^2$ , we cannot have  $p = 2$  or  $3$ . Conditions (Ia,b) become  $3a_0 - 2Q \equiv 0$ ,  $2a_1Q + 3R \equiv 0 \pmod{p^n}$ , where  $p^{2n}/\Delta$ . Substituting from these in the congruences,  $\pmod{p^{2n}}$  and  $\pmod{p^{3n}}$ , derived from (II) and (III), we may omit terms congruent to 0 with these moduli, even if divided by  $2, 3, Q$ , leaving  $(II') \Delta \equiv 0 \pmod{p^{2n}}$ , and  $(III') \Delta^2 \equiv 0 \pmod{p^{3n}}$ , respectively. Since  $p^{2n}/\Delta$ , these are satisfied.

There remain only the cases in which  $p = 2$  and  $2/R$  but not  $2/Q$ , and in which  $p = 3$  and  $3/Q$  but not  $3/R$ . In the former, from Corollary 3, Theorem V, we must have an integer of the forms  $(\frac{1}{2}, 0, \frac{1}{2})$  or  $(0, \frac{1}{2}, \frac{1}{2})$ . Only

the latter satisfies (Ia), while (II) requires that  $R \equiv Q + 1 \pmod{4}$ . If this integer is not maximal, the latter must be of the form  $(a_0 \div 2^t, a_1 \div 2^n, 1 \div 2^n)$ , where  $a_1 \equiv 1 \pmod{2}$ , and  $2^{2n}/\Delta = -4Q^3 - 27R^2$ . Since  $Q$  is odd,  $R \equiv 2 \pmod{4}$ , from the last, and  $Q \equiv 1 \pmod{4}$ . From (Ia,b),  $t = n - 1$ ,  $3a_0 - Q = 2^{n-1}a$ ,  $2a_1Q + 3R = 2^n b$ . Writing  $\Delta = 2^{2n}\Delta'$ , from (II) and (III) we obtain (II')  $b^2 \equiv \Delta' \pmod{4}$ , and (III')  $6Q^2a(3b^2 + \Delta') - 3Rb(b^2 - \Delta') + 2^n\Delta'(\Delta' - b^2) \equiv 0 \pmod{8}$ . Hence, if  $b$  is odd,  $\Delta' \equiv 1 \pmod{4}$ , and, if  $b$  is even,  $\Delta' \equiv 0 \pmod{4}$ , or  $2^{2n+2}/\Delta$ . Hence, if  $\Delta$  is of the form  $2^{2k}(4s + 1)$ , we have  $n = k$  for the maximal, and  $2a_1Q + 3R \equiv 0 \pmod{2^n}$  but  $\not\equiv 0 \pmod{2^{n+1}}$ ; otherwise,  $n$  is the greatest index such that  $2^{2n+2}/\Delta$ , and  $2a_1Q + 3R \equiv 0 \pmod{2^{n+1}}$ .

If  $p = 3$  and  $3/Q$  but not  $3/R$ , in the integer referred to in Theorem V,  $a_0 \neq 0$ ,  $a_1 \neq 0$  (Corollary 4), and, if  $y = \frac{1}{3}(a_0, a_1, 1)$ ,

$$yx = \left( -\frac{R}{3}, \frac{a_0 - Q}{3}, \frac{a_1}{3} \right) = a_1 y,$$

apart from ordinary integers. Hence  $a_0 a_1 \equiv -R$ ,  $a_1^2 \equiv a_0 \pmod{3}$ , giving  $a_0 = 1$ ,  $a_1 = \pm 1$  since  $-\frac{1}{2} < (a \div 3) \leq \frac{1}{2}$ . From (III),  $R \equiv \pm(Q + 1) \pmod{9}$ , according as  $a_1 = \mp 1$ , and (Ia, II) are satisfied, giving the integer  $\frac{1}{3}(1, \mp 1, 1)$ . This is maximal unless there is a maximal reduced integer of degree 1 in  $x$ . For, if not and  $y = \frac{1}{3}(a_0, a_1, 1)$ , then  $3y = (1, \mp 1, 1)$ , apart from ordinary integers, giving  $a_0 \equiv 1$ ,  $a_1 \equiv \pm 1 \pmod{3}$ . From (Ia),  $Q \equiv -3 \pmod{9}$ . As above,  $a_0 a_1 \equiv \pm 2$ ,  $a_0 + 3 \equiv a_1^2 \pmod{9}$ , from which we have  $(a_0, a_1) = (1, \pm 2), (4, \mp 4), (-2, \mp 1)$ . Substituting in (III), we obtain either a contradiction of  $R \equiv \pm(Q + 1) \pmod{9}$ , or the condition  $R \equiv \pm(Q + 1) \pmod{27}$ , the condition already obtained for the existence of a maximal reduced integer of degree 1.

If there is such an integer, viz.  $\frac{1}{3}(\pm 1, 1, 0)$ , its square,  $\frac{1}{9}(1, \pm 2, 1)$  is an integer. Since  $Q \equiv -3 \pmod{9}$ , and  $R \not\equiv 0 \pmod{3}$ , from (Ia,b), an integer of higher degree in 3 must be homogeneous in 3. If  $(1 \div 3^n)(a_0, a_1, 1)$  is this integer, from (Ia,b),  $3a_0 - 2Q = 3^n a$ ; also  $2a_1Q + 3R = 3^n b$ , and from Theorem III,  $3^{2n+2}/\Delta$ . Writing  $\Delta = 3^{2n+2}\Delta'$  and substituting in (II) and (III), we obtain (II')  $a^2 \equiv b^2 \pmod{3}$ , and (III')  $8Q^3 a^3 + 18Q^2 ab^2 - 27Rb^3 + 54Q^2 \Delta' a + 243R \Delta' b \equiv 0 \pmod{3^6}$ . From the latter, since  $Q \equiv -3 \pmod{9}$ ,  $a^3 \equiv Rb^3 \pmod{3}$ , or  $a \equiv \pm b \pmod{3}$  according as  $R \equiv \mp 2 \pmod{9}$ . The solutions  $a \equiv \pm b \equiv \pm 1 \pmod{3}$  evidently lead to values of

$a_0, a_1$ , differing from those obtained from  $a \equiv b \equiv 0 \pmod{3}$  by  $3^{n-1}$ ; i.e. the integers obtained differ by  $\frac{1}{3}(x \pm 1)$ . As any one of the three will serve for the maximal reduced integer (Theorem IIa), we may take  $a \equiv b \equiv 0 \pmod{3}$ , and both (II') and (III') are satisfied. The maximal reduced integer is therefore  $(1 \div 3^n)(a_0, a_1, 1)$ , where  $a_0$  and  $a_1$  are determined from  $3a_0 - 2Q \equiv 0$ ,  $2a_1Q + 3R \equiv 0 \pmod{3^{n+1}}$ ,  $n$  being the greatest index such that  $3^{2n+2}/\Delta$ .

#### BASIS FOR A GENERAL CUBIC

For the general cubic,  $A_0z^3 + A_1z^2 + A_2z + A_3 = 0$ , all  $A$ 's bring rational integers, we obtain  $x^3 + Qx + R = 0$  in the normal form by writing  $dx = A_0z + A_1$ ,  $Q = 3(A_0A_2 - A_1) \div d^2$ ,  $R = (A_0A_3 - 3A_0A_1A_2 + 2A_1^2) \div d^3$ , where  $d$  is the greatest rational integer for which such division is possible; also  $\Delta = -4Q^3 - 27R^2$ . Then 1 is one element of the basis;  $x = (A_0z + A_1) \div d$  is a second element unless  $Q \equiv -3 \pmod{9}$ ,  $R \equiv \pm(Q+1) \pmod{27}$ , when  $\frac{1}{3}(x \pm 1) = (A_0z + A_1 \pm d) \div 3d$  is the second element. To determine the third element, the complete set of maximal reduced integers  $m_i$  of degrees  $n_i$  in the primes  $p_i$ , where  $p_i^2$  is a factor of  $\Delta$ , is determined from the table below. The third element is  $\sum u_i m_i$ , where the  $u$ 's are rational integral non-zero solutions of  $\sum (u_i \div p_i^{n_i}) = 1$ . The solutions  $a_0$  and  $a_1$  of the congruences below are such that  $-\frac{1}{2} < (a \div D) \leq \frac{1}{2}$ , where  $D$  is the denominator given.

	Max. Red. Int.	Conditions
1	$\frac{x^2}{p}$	$p/Q$ and $p^2/R$ ; $n = 1$ .
2	$\frac{a_0 + a_1x + x^2}{p^n}$	$p$ prime to $6QR$ ; $n$ greatest index such that $p^{2n}/\Delta$ ; $3a_0 - 2Q \equiv 0$ , $2a_1Q + 3R \equiv 0 \pmod{p^n}$ .
3a	$\frac{x + x^2}{2}$	$p = 2, Q \equiv 1 \pmod{2}, R \equiv Q + 1 \pmod{4}$ , (3b) unsatisfied; $n = 1$ .
3b	$\frac{a_0}{2^n} + \frac{a_1}{2^n}x + \frac{1}{2^n}x^2$	$p = 2, Q \equiv 1, R \equiv 2 \pmod{4}$ ; if $\Delta$ is of the form $2^{2k}(4s+1)$ , $n = k$ , $2a_1Q + 3R \equiv 0 \pmod{2^n}$ , but $\not\equiv 0 \pmod{2^{n+1}}$ ; otherwise, $n$ is the greatest index such that $2^{2n+2}/\Delta, 2a_1Q + 3R \equiv 0 \pmod{2^{n+1}}$ ; $3a_0 - Q \equiv 0 \pmod{2^{n-1}}$ .
4a	$\frac{1 + x + x^2}{3}$	$p = 3, Q \equiv 0 \pmod{3}, R \equiv \pm(Q+1) \pmod{9}$ , (4b) unsatisfied; $n = 1$ .
4b	$\frac{1 + 2x + x^2}{9}$	$p = 3, Q \equiv -3 \pmod{9}, R \equiv \pm(Q+1) \pmod{27}$ , (4c) unsatisfied; $n = 2$ .
4c	$\frac{a_0 + a_1x + x^2}{3}$	$p = 3, 3^{2n+2}/\Delta, n > 2; 3a_0 - 2Q \equiv 0, 2a_1Q + 3R \equiv 0 \pmod{3^{n+1}}$ , where $n$ is the greatest index such that $3^{2n+2}/\Delta$ .

UNIVERSITY OF MANITOBA,  
WINNIPEG, CANADA

# IMPLICIT FUNCTIONS AND THEIR DIFFERENTIALS IN GENERAL ANALYSIS\*

BY

T. H. HILDEBRANDT

AND

LAWRENCE M. GRAVES†

**Introduction.** Implicit function theorems occur in analysis in many different forms, and have a fundamental importance. Besides the classical theorems as given for example in Goursat's *Cours d'Analyse* and Bliss's *Princeton Colloquium Lectures*, and the classical theorems on linear integral equations, implicit function theorems in the domain of infinitely many variables have been developed by Volterra, Evans, Lévy, W. L. Hart and others.‡

The existence and imbedding theorems for solutions of differential equations, as treated for example by Bliss,§ have also received extensions to domains of infinitely many variables by Moulton, Hart, Barnett, Bliss and others.¶ Special properties in the case of linear differential equations in an infinitude of variables have been treated by Hart and Hildebrandt.|| On the other hand, Hahn and Carathéodory\*\* have made important generalizations of the notion of differential equation by removing continuity restrictions on the derivatives and by writing the equations in the form of integral equations.

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†National Research Fellow in Mathematics.

‡See Volterra, *Fonctions de Lignes*, Chapter 4; Evans, *Cambridge Colloquium Lectures*, pp. 52-72; Lévy, *Bulletin de la Société Mathématique de France*, vol. 48 (1920), p. 13, Hart, these *Transactions*, vol. 18 (1917), pp. 125-150, where additional references may be found; also vol. 23 (1922), pp. 45-50, and *Annals of Mathematics*, (2), vol. 24 (1922) pp. 29 and 35.

§ *Princeton Colloquium*, and *Bulletin of the American Mathematical Society*, vol. 25 (1918), p. 15.

¶ Moulton, *Proceedings of the National Academy of Sciences*, vol. 1 (1915), p. 350; Hart, papers cited above; Barnett, *American Journal of Mathematics*, vol. 44 (1922), p. 172; Bliss, these *Transactions*, vol. 21 (1920), p. 79.

|| Hart, *American Journal of Mathematics*, vol. 39 (1917), p. 407; Hildebrandt, these *Transactions*, vol. 18 (1917), p. 73, and vol. 19 (1918), p. 97.

\*\* Hahn, *Monatshefte für Mathematik und Physik*, vol. 14 (1903), p. 326; Carathéodory, *Vorlesungen über Reelle Funktionen*, p. 666.

These various theories suggest the desirability of a general unifying theory. Lamson began such a general theory in his paper entitled *A general implicit function theorem*.<sup>\*</sup> However, his theory is limited in power, because it contains no theorems on the differentiability of the solutions. Most of the papers on special cases by the other writers mentioned above have a like defect, in that they give, if any, a very incomplete theory of differentiability properties.

The authors propose in the present paper to give a simple general theory of implicit functions and their differentiability, and in a second paper to develop a few special cases, including some new results which find application in the calculus of variations.

The classical implicit function theorems deal with the solution of equations  $G(x, y) = 0$ , where  $x$  and  $y$  are points of ordinary space of one or more dimensions. In this paper the authors generalize the theory to the case where  $x$  and  $y$  are points of abstract spaces of the type discussed by Fréchet. However, no previous knowledge of Fréchet's theory is needed or used. Also the notation is so devised that the reader may readily compare the theory with the classical theory of implicit functions, and interpret it in the finite domain if he so desires.

In the Part I the fundamental postulates and definitions are set down, and certain fundamental propositions are proved. Part II contains the preliminary theorems on the solution of equations in the form  $y = F(x, y)$ . In Part III, the differential calculus of functions in our abstract spaces is developed. This depends on the notion of total differential, as defined by Stolz<sup>†</sup> for the case of an ordinary function of  $n$  variables, applied by Fréchet<sup>‡</sup> in the theory of functionals, and finally developed by Fréchet (independently of the authors of this paper) for the general case.<sup>§</sup> Part IV contains lemmas concerning reciprocal linear functions, and Part V contains the final theorems on the existence and differentiability of implicit functions defined by equations of the form

$$G(x, y) = y_*$$

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<sup>\*</sup> American Journal of Mathematics, vol. 42 (1920), p. 243.

<sup>†</sup> *Grundzüge der Differential- und Integralrechnung*, 1893, vol. 1, pp. 130 ff., 155 ff. Cf. also W. H. Young, Proceedings of the London Mathematical Society, vol. 7 (1909), p. 157; and Fréchet, *Sur la notion de différentielle totale*, *Comptes Rendus du Congrès des Sociétés Savantes en 1914, Sciences*.

<sup>‡</sup> These Transactions, vol. 15 (1914), p. 140.

<sup>§</sup> *Comptes Rendus*, vol. 180 (1925), p. 806, and *Annales de l'École Normale Supérieure*, vol. 41 (1925), p. 293.

## I. POSTULATES, DEFINITIONS, AND FUNDAMENTAL PROPOSITIONS

1. **Postulates.** We shall be dealing in the following sections with systems  $(\mathfrak{X}, \|\cdot\|, \oplus, \odot)$  having some or all of the following properties.\*

1.11.  $\mathfrak{X}$  is either the real number system or the complex number system.

1.12.  $\mathfrak{X}$  is a class having at least two distinct elements.

1.13.  $\|\cdot\|$  is a function on  $\mathfrak{X}\mathfrak{X}$  to the real non-negative part of  $\mathfrak{X}$ , i. e., to every pair  $x_1, x_2$  of elements of  $\mathfrak{X}$  corresponds a unique positive or zero number denoted by  $\|x_1, x_2\|$ .

1.14.  $\|x_1, x_2\| = \|x_2, x_1\|$  for every  $x_1, x_2$ .

1.15.  $\|x_1, x_2\| = 0$  is equivalent to  $x_1 = x_2$ .

1.16.  $\|x_1, x_2\| \leq \|x_1, x_3\| + \|x_3, x_2\|$  for every  $x_1, x_2, x_3$ .

1.2. For every sequence  $\{x_n\}$  such that

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|x_m, x_n\| = 0$$

there exists an element  $x$  such that

$$\lim_{n \rightarrow \infty} \|x_n, x\| = 0.$$

1.31.  $\oplus$  is a function on  $\mathfrak{X}\mathfrak{X}$  to  $\mathfrak{X}$ , i. e., to every pair  $x_1, x_2$  there corresponds a unique element of  $\mathfrak{X}$  denoted by  $x_1 \oplus x_2$  and called the sum of  $x_1$  and  $x_2$ .

1.32.  $\oplus$  is commutative, i. e.,  $x_1 \oplus x_2 = x_2 \oplus x_1$  for every  $x_1, x_2$ .

1.33.  $\oplus$  is associative, i. e.,  $(x_1 \oplus x_2) \oplus x_3 = x_1 \oplus (x_2 \oplus x_3)$  for every  $x_1, x_2, x_3$ .

1.34.  $\odot$  is a function on  $\mathfrak{X}\mathfrak{X}$  to  $\mathfrak{X}$ , i. e., to every element  $x$  and number  $a$  there corresponds a unique element of  $\mathfrak{X}$ , denoted by  $x \odot a$ , and called the product of  $x$  by  $a$ .

1.35.  $\odot$  is associative, i. e.,  $(x \odot a_1) \odot a_2 = x \odot (a_1 a_2)$  for every  $x, a_1, a_2$ .

1.36.  $\odot$  is doubly distributive, i. e.,  $(x_1 \oplus x_2) \odot a = x_1 \odot a \oplus x_2 \odot a$ , and  $x \odot (a_1 + a_2) = x \odot a_1 \oplus x \odot a_2$ , for every  $x_1, x_2, x, a, a_1, a_2$ .

1.37.  $x \odot 1 = x$  for every  $x$ .

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\* For similar sets of postulates, cf. Banach, *Fundamenta Mathematicae*, vol. 3 (1922), p. 133; Hahn, *Monatshefte für Mathematik und Physik*, vol. 32 (1922), p. 3; Fréchet, *Comptes Rendus*, vol. 180 (1925), p. 419. The sets of Banach and of Hahn are somewhat redundant. Our thanks are due to M. H. Ingraham for suggestions tending to eliminate some superfluous postulates.

**Notation.** The element  $x \odot (-1)$  will be denoted by  $-x$ , and the element  $x_1 \oplus (-x_2)$  will be denoted by  $x_1 - x_2$ . The number  $\|x, x \odot 0\|$  will be denoted by  $\|x\|$ . When no ambiguity can arise, the sign  $\oplus$  will be replaced by the ordinary  $+$ , and  $\odot$  will be omitted altogether.

1.38.  $\|x_1, x_2\| = \|x_1 - x_2\|$  for every  $x_1, x_2$ .

1.39.  $\|xa\| = \|x\| |a|$  for every  $x$  and  $a$ , where  $|a|$  denotes the ordinary absolute value of  $a$ .

If a system  $(\mathfrak{X}, \mathfrak{N}, \|\cdot\|, \oplus, \odot)$  satisfies the group of postulates 1.11 to 1.16, we shall call  $\mathfrak{X}$  a *metric space*; if 1.11 to 1.16 and 1.31 to 1.39 are satisfied we shall call  $\mathfrak{X}$  a *linear metric space*; and in either case if 1.2 is also satisfied we shall say that  $\mathfrak{X}$  is *complete*.\* The elements of a metric space will be called *points*. If we denote the real number system by  $\mathfrak{R}$  and the complex number system by  $\mathfrak{C}$ , we readily verify that the systems

$$\mathfrak{X} = \mathfrak{R}, \mathfrak{N} = \mathfrak{R}, \|r_1, r_2\| = |r_1 - r_2|, r_1 \oplus r_2 = r_1 + r_2, r_1 \odot r = r_1 r,$$

and

$$\mathfrak{X} = \mathfrak{C}, \mathfrak{N} = \mathfrak{C}, \|c_1, c_2\| = |c_1 - c_2|, c_1 \oplus c_2 = c_1 + c_2, c_1 \odot c = c_1 c,$$

satisfy all the postulates 1.11 to 1.39. Moreover, if  $\mathfrak{X}$  is linear metric with  $\mathfrak{C}$  as its associated number system, it is also linear metric with  $\mathfrak{R}$  as its associated number system.

**2. Properties of spaces.** A metric space  $\mathfrak{X}$  has the following additional properties.

2.11. If the sequence  $\{x_n\}$ , the points  $x$  and  $x'$ , and the number  $a$  are such that

$$\lim_{n \rightarrow \infty} \|x_n, x\| = 0,$$

and  $\|x_n, x'\| \leq a$  for every  $n$ , then  $\|x, x'\| \leq a$ .

2.12. If a sequence  $\{x_n\}$  has a limit, it has only one, i. e., if  $\{x_n\}$ ,  $x_1$  and  $x_2$  are such that

$$\lim_{n \rightarrow \infty} \|x_n, x_1\| = 0, \quad \lim_{n \rightarrow \infty} \|x_n, x_2\| = 0,$$

then  $x_1 = x_2$ .

The first of these follows from 1.14 and 1.16, and the second from 1.14, 1.16 and 1.15.

For a complete metric space  $\mathfrak{X}$  we have the following proposition:

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\*A metric space is one of the classes denoted by  $(\mathfrak{E})$  in Fréchet's thesis, and by  $(\mathfrak{D})$  in his later work. A complete linear metric space is called by Fréchet a space of Banach, or "espace  $(\mathfrak{D})$  vectoriel complet", and is a special case of his "espace affine" for which the postulates are set down in the note cited above.

2.21. If the sequence  $\{x_n\}$  is such that the series

$$\sum_n \|x_n, x_{n+1}\|$$

is convergent, then the sequence has a unique limit.

This follows from 1.14, 1.16, 1.2, and 2.12.

A linear metric space  $\mathfrak{X}$  has the following additional properties.

2.31.  $x_1 \odot 0 = x_2 \odot 0$  for every  $x_1, x_2$ .

Notation. The point  $x \odot 0$ , which is independent of  $x$  by 2.31, will be denoted by  $x_*$ .

2.32.  $x + x_* = x$  for every  $x$ , and  $x_*$  is the only point of  $\mathfrak{X}$  having this property.

2.33.  $xa = x_*$  is equivalent to  $a = 0$  or  $x = x_*$ .

2.34. For every pair  $x_1, x_2$ , there exists uniquely a point  $x_3$  such that  $x_1 + x_3 = x_2$ , viz.,  $x_3 = x_2 - x_1$ .

2.35. If  $a \neq 0$  and  $x_1 a = x_2 a$ , then  $x_1 = x_2$ .

2.36. If  $x \neq x_*$  and  $xa_1 = xa_2$ , then  $a_1 = a_2$ .

2.37.  $||x_1| - |x_2|| \leq ||x_1 \pm x_2|| \leq ||x_1|| + ||x_2||$  for every  $x_1, x_2$ .

The property 2.31 follows from 1.11, 1.34, 1.38, 1.35, 1.36, 1.39, and 1.15; 2.32 from 1.11, 2.31, 1.37, 1.36, 1.32; 2.33 from 1.11, 2.31, 1.34, 1.35, 1.37; 2.34 from 1.11, 1.31, 1.32, 1.33, 1.37, 1.36, 2.31, 2.32; 2.35 from 1.11, 1.34, 1.35, 1.37; 2.36 from 1.11, 1.31, 1.36, 2.31, 2.33; 2.37 from 1.11, 1.38, 2.31, 1.14, 1.16, 1.31, 1.32, 1.35, 1.36, 1.37, 1.39, 2.32, 2.34.

3. Composition of classes. For definiteness consider two classes  $\mathfrak{X}$  and  $\mathfrak{Y}$ . Then the composite class  $\mathfrak{B} = (\mathfrak{X}, \mathfrak{Y})$  is defined to be the class of all pairs  $(x, y)$  of elements, one from  $\mathfrak{X}$  and one from  $\mathfrak{Y}$ . If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are metric spaces, and if we define

$$(3.1) \quad \|w_1, w_2\| \equiv \|(x_1, y_1), (x_2, y_2)\| \equiv \text{greater of } \begin{cases} a\|x_1, x_2\| \\ b\|y_1, y_2\| \end{cases}$$

where  $a$  and  $b$  are fixed positive constants, then  $\mathfrak{B}$  is also a metric space.\* Unless otherwise specified in the sequel, we take  $a = b = 1$ . If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are complete metric spaces, so is  $\mathfrak{B}$ . If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are linear metric spaces with the same associated number system  $\mathfrak{A}$ , and if we define  $w_1 + w_2 \equiv (x_1, y_1) + (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2)$ ,  $wa \equiv (x, y)a \equiv (xa, ya)$ , then  $\mathfrak{B}$  is also a linear metric space. We note that any number of classes may be composed in this way, and that composition may be regarded as an associative process.

\*Of course many other definitions might be used for  $\|w_1, w_2\|$ , e. g.,  $\|w_1, w_2\| = (\|x_1, x_2\|^p + \|y_1, y_2\|^p)^{1/p}$  ( $p > 1$ ). The one given in the text seems to be the most convenient for our purposes.

4. **Neighborhoods.** For a given point  $x_0$  of a metric space  $\mathfrak{X}$ , and a positive number  $a$ , we define the neighborhood  $(x_0)_a$  to be the set of all points  $x$  such that  $\|x, x_0\| < a$ . When we speak of "a neighborhood" of a point  $x$ , we shall always mean a neighborhood of this type. A neighborhood of  $x$  may contain no points distinct from  $x$ , unless  $\mathfrak{X}$  is linear.

5. **Regions.** A region of a metric space  $\mathfrak{X}$  is a set  $\mathfrak{X}_0$  of points of  $\mathfrak{X}$  such that every point  $x$  of  $\mathfrak{X}_0$  has a neighborhood consisting wholly of points of  $\mathfrak{X}_0$ . Regions will be consistently designated by attaching subscripts to the German capitals representing the spaces to which they belong. A neighborhood of a point  $x$  is an example of a region. In a composite space  $(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{B}$ , a neighborhood  $(w_0)_c = ((x_0, y_0))_c$  consists of all pairs  $(x, y)$  for which  $x$  is in  $(x_0)_d$  and  $y$  is in  $(y_0)_e$ , where  $d = c/a$ ,  $e = c/b$ , and  $a$  and  $b$  are the constants of the definition (3.1) above. The composite  $\mathfrak{B}_0 = (\mathfrak{X}_0, \mathfrak{Y}_0)$  of two regions  $\mathfrak{X}_0$  and  $\mathfrak{Y}_0$  is also a region in the composite space  $\mathfrak{B} = (\mathfrak{X}, \mathfrak{Y})$ , but the converse need not be true. However, the set  $\mathfrak{Y}_0$  of points of  $\mathfrak{Y}$  such that  $(x_0, y)$  belongs to a region  $\mathfrak{B}_0$  of  $\mathfrak{B}$  constitutes a region for each fixed  $x_0$  of  $\mathfrak{X}$ , unless the set is empty.

6. **Continuous functions. Relative uniformity.** We shall have frequent use for the notion of relative uniformity, due to E. H. Moore.\* Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be two metric spaces, and let  $\mathfrak{P}$  be a general range. Let  $F$  be a function on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$ , i. e.,  $F$  makes correspond to each point  $x$  of the region  $\mathfrak{X}_0$  of  $\mathfrak{X}$  and each element  $p$  of the class  $\mathfrak{P}$  one and only one element  $F(x, p)$  of  $\mathfrak{Y}$ . Let  $\sigma$  be a function on  $\mathfrak{P}$  to  $\mathfrak{A}$ . Then we say that  $F$  is continuous at a point  $x_0$  of  $\mathfrak{X}_0$  uniformly on  $\mathfrak{P}$  relative to  $\sigma$ , more briefly, *uniformly*  $(\mathfrak{P}; \sigma)$ , in case for every  $\epsilon > 0$  there exists a  $d > 0$  such that, for every  $x$  in  $(x_0)_d$  and every  $p$  of  $\mathfrak{P}$ , we have  $\|F(x, p), F(x_0, p)\| \leq \epsilon|\sigma(p)|$ . We say that  $F$  is continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$  in case  $F$  is continuous at each point of  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$ . Obviously we obtain the definition of ordinary continuity as a special case by taking a singular range  $\mathfrak{P}$  and  $\sigma(p) = 1$ . We say that  $F$  is continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{X}_0\mathfrak{P}; \sigma)$  in case for every  $\epsilon > 0$  there exists a  $d > 0$  such that, for every  $x_1$  and  $x_2$  of  $\mathfrak{X}_0$  satisfying  $\|x_1, x_2\| \leq d$  and for every  $p$  of  $\mathfrak{P}$  we have  $\|F(x_1, p), F(x_2, p)\| \leq \epsilon|\sigma(p)|$ .

Note that it is important to specify the range of uniformity. In the applications we shall make of the notion of relative uniformity, the range  $\mathfrak{P}$  will frequently have as a component a linear metric space  $\mathfrak{B}$ , and the scale function  $\sigma$  will then have as a factor the norm  $\|w\|$ .

\* Cf. *Introduction to a Form of General Analysis*, p. 27.

7. **Connected sets.** If  $\mathfrak{X}$  is a metric space, we say that a set  $\mathfrak{X}^{(0)}$  of points of  $\mathfrak{X}$  is a connected set in case for every pair  $x_1, x_2$  of points of  $\mathfrak{X}^{(0)}$  there exists a function  $F$  on an interval  $\mathfrak{N}_0$  (where  $\mathfrak{N}$  is the axis of reals) to  $\mathfrak{X}^{(0)}$  which is continuous on  $\mathfrak{N}_0$  and such that  $F(r_1) = x_1, F(r_2) = x_2$ , where  $r_1$  and  $r_2$  are points of  $\mathfrak{N}_0$ . Not every metric space contains a connected set. However, in a linear metric space, a neighborhood forms an example of a connected region. The composite  $(\mathfrak{X}^{(0)}, \mathfrak{Y}^{(0)})$  of two connected sets  $\mathfrak{X}^{(0)}$  and  $\mathfrak{Y}^{(0)}$  is also a connected set.

8. **Boundary of a set.** We shall say that a point  $x$  belongs to the boundary of a set  $\mathfrak{X}^{(0)}$  of a metric space  $\mathfrak{X}$  in case every neighborhood of  $x$  contains both a point of  $\mathfrak{X}^{(0)}$  and a point not of  $\mathfrak{X}^{(0)}$ . The boundary of  $\mathfrak{X}^{(0)}$  consists of all such points. By definition of a region, no point of the boundary of a region  $\mathfrak{X}_0$  belongs to  $\mathfrak{X}_0$ . Also if every neighborhood of a point  $x$  contains a point of a set  $\mathfrak{X}^{(0)}$ , then  $x$  belongs either to  $\mathfrak{X}^{(0)}$  or to the boundary of  $\mathfrak{X}^{(0)}$ .

## II. EXISTENCE AND CONTINUITY OF SOLUTIONS OF EQUATIONS OF THE FORM $y = F(x, y)$

9. Throughout this section we shall denote by  $\mathfrak{Y}$  a complete metric space, and by  $\mathfrak{X}$  a metric space.

**THEOREM 1.** *Let the point  $y_0$  of  $\mathfrak{Y}$  and the region  $\mathfrak{X}_0$  of  $\mathfrak{X}$  and the function  $F$  on  $\mathfrak{X}_0(y_0)_a$  to  $\mathfrak{Y}$  be such that*

*(H<sub>1</sub>) for every  $x$  in  $\mathfrak{X}_0$  there exists a positive constant  $k_x < 1$  such that*

$$\|F(x, y_1), F(x, y_2)\| \leq k_x \|y_1, y_2\|$$

*for every pair  $y_1, y_2$  in  $(y_0)_a$ ;*

$$(H_2) \quad \|F(x, y_0), y_0\| < (1 - k_x)a$$

*for every  $x$  in  $\mathfrak{X}_0$ . Then there exists a unique function  $Y$  on  $\mathfrak{X}_0$  to  $(y_0)_a$  such that*

$$(9.1) \quad Y(x) = F(x, Y(x))$$

*for every  $x$  in  $\mathfrak{X}_0$ .*

This theorem is basic for all the following theorems, and its proof is the only place where a sequence of approximating functions is used. In most applications of this theorem the constant  $k_x$  may be taken independent of  $x$ . It should be noted also that only an approximate initial solution is required. The proof is as follows.

We define a sequence of approximations by the equations

$$(9.2) \quad Y_1(x) = F(x, y_0), \quad Y_{m+1}(x) = F(x, Y_m(x)) \quad (m > 0).$$

By  $H_2$  we have  $\|Y_1(x), y_0\| = (1 - k_x)c_x$ , where  $c_x < a$ , and by  $H_1$  and induction,

$$(9.3) \quad \|Y_{m+1}(x), Y_m(x)\| \leq k_x^m(1 - k_x)c_x \quad (m > 0).$$

From this we obtain

$$(9.4) \quad \|Y_m(x), y_0\| < c_x < a \quad (m > 0),$$

so that the approximations (9.2) are surely defined when  $x$  is in  $\mathfrak{X}_0$ .

From the inequality (9.3) and property 2.21 we conclude that there exists a unique function  $Y$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  such that

$$(9.5) \quad \lim_{m \rightarrow \infty} \|Y_m(x), Y(x)\| = 0.$$

That  $Y(x)$  is in  $(y_0)_a$  follows from the inequality (9.4) and property 2.11. By  $H_1$  we have

$$\begin{aligned} \|Y, F(x, Y)\| &\leq \|Y, Y_{m+1}\| + \|F(x, Y_m), F(x, Y)\| \\ &\leq \|Y, Y_{m+1}\| + k_x \|Y_m, Y\|, \end{aligned}$$

from which by equation (9.5) and property 1.15 we obtain the desired equation (9.1). To obtain the uniqueness of the solution we again apply  $H_1$  and property 1.15.

We note that the hypotheses of Theorem 1 imply that for each  $x$  of the region  $\mathfrak{X}_0$  the function  $F(x, y)$  transforms the neighborhood  $(y_0)_a$  into a part of itself. For each  $x$  the solution  $Y(x)$  is the unique invariant point of this transformation. This point of view has been developed by Birkhoff and Kellogg\* and by J. L. Holley† for certain special cases.

**THEOREM 2.** Let  $\mathfrak{B}_0$  be a region of the composite space  $(\mathfrak{X}, \mathfrak{Y})$ , and let the function  $F$  on  $\mathfrak{B}_0$  to  $\mathfrak{Y}$  and the point  $(x_0, y_0)$  of  $\mathfrak{B}_0$  be such that

$$(H_1) \quad y_0 = F(x_0, y_0);$$

$$(H_2) \quad \text{there exists a positive constant } k < 1 \text{ such that}$$

$$\|F(x, y_1), F(x, y_2)\| \leq k \|y_1, y_2\|$$

for every  $(x, y_1), (x, y_2)$  in  $\mathfrak{B}_0$ ;

( $H_3$ )  $F$  is continuous in its argument  $x$  at  $(x_0, y_0)$ . Then the following conclusions hold:

( $C_1$ ) for each  $x$  there is at most one point  $(x, y)$  in  $\mathfrak{B}_0$  which is a solution of the equation.

$$(9.6) \quad y = F(x, y);$$

\* These Transactions, vol. 23 (1922), p. 96.

†Harvard thesis, 1924.

(C<sub>2</sub>) there exist a region  $\mathfrak{X}_1$  containing  $x_0$  and a function  $Y$  on  $\mathfrak{X}_1$  to  $\mathfrak{Y}$  such that the point  $(x, Y(x))$  is in  $\mathfrak{B}_0$  and is a solution of (9.6) for every  $x$  in  $\mathfrak{X}_1$ ;

(C<sub>3</sub>) the solution  $Y$  is continuous at  $x_0$ ;

(C<sub>4</sub>) if  $F$  is continuous on  $\mathfrak{B}_0$  [uniformly  $(\mathfrak{B}_0; 1)$ , and if  $\mathfrak{B}_0$  is the composite of two regions  $\mathfrak{X}_0$  and  $\mathfrak{Y}_0$ ], then the solution  $Y$  of equation (9.6) is continuous on its domain of definition [uniformly on that domain].

$C_1$  follows as before from  $H_2$  and Postulate 1.15. By the definition of region, there exist positive constants  $a$  and  $b$  such that the region  $((x_0)_b, (y_0)_a)$  is contained in  $\mathfrak{B}_0$ . By  $H_1$  and  $H_3$ , if  $b$  is sufficiently small,  $H_2$  of Theorem 1 is satisfied on  $(x_0)_b$ . Thus  $C_2$  follows from Theorem 1. To obtain  $C_3$  and  $C_4$  (where the parts in brackets form an alternative reading), we use the inequalities

$$\begin{aligned} \|Y_1, Y_2\| &\leq \|F(x_1, Y_1), F(x_1, Y_2)\| + \|F(x_1, Y_2), F(x_2, Y_2)\| \\ &\leq k\|Y_1, Y_2\| + \|F(x_1, Y_2), F(x_2, Y_2)\|, \\ \|Y_1, Y_2\| &\leq (1/(1-k)) \|F(x_1, Y_2), F(x_2, Y_2)\|, \end{aligned}$$

where we have set  $Y(x_1) = Y_1$ ,  $Y(x_2) = Y_2$ . These inequalities are valid in all cases, at least if  $x_1$  is in a sufficiently small neighborhood of  $x_2$ .

Note that  $C_1, C_2, C_4$  are still valid if in place of an exact initial solution as assumed in  $H_1$ , we have merely an approximate solution, as in Theorem 1, i. e., if merely  $\|F(x_0, y_0), y_0\|$  is sufficiently small.

### III. DIFFERENTIALS. THE CLASS $\mathfrak{C}^{(n)}$

Throughout this section and the succeeding ones, we shall assume that we are working with *linear* metric spaces  $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \dots$ . In addition we shall frequently add a general range  $\mathfrak{P}$ . The associated number system is assumed to be the same for all spaces considered.

10. **Linear functions.** We say that the function  $F$  on  $\mathfrak{X}\mathfrak{P}$  to  $\mathfrak{Y}$  is *distributive* on  $\mathfrak{X}$  if for every  $a_1, a_2, x_1, x_2$  and  $p$  it is true that

$$F(a_1x_1 + a_2x_2, p) = a_1F(x_1, p) + a_2F(x_2, p)^*.$$

The function  $F$  on  $\mathfrak{X}\mathfrak{P}$  to  $\mathfrak{Y}$  is said to be *modular on  $\mathfrak{X}$  uniformly on  $\mathfrak{P}$  relative to  $\sigma$*  (or uniformly  $(\mathfrak{P}; \sigma)$ ) in case there exists a constant  $M$  such that for every  $x$  and  $p$ ,

$$\|F(x, p)\| \leq M\|x\| \cdot |\sigma(p)|.$$

\*We shall frequently omit the argument  $p$  in equations similar to this one especially when  $p$  enters "homogeneously."

The minimum of effective values for  $M$ , which minimum obviously exists, is called the *modulus* of  $F$ .

The function  $F$  on  $\mathfrak{X}$  to  $\mathfrak{Y}$  is said to be *linear* on  $\mathfrak{X}$  uniformly  $(\mathfrak{P}; \sigma)$  if it is distributive and modular uniformly  $(\mathfrak{P}; \sigma)$ .

With respect to functions having the modular property we have

**LEMMA 10.1.** *If  $F$  on  $\mathfrak{X}_0 \mathfrak{Y}$  to  $\mathfrak{Z}$  is continuous at  $x_0$  uniformly  $(\mathfrak{Y}; \|\cdot\| \sigma)$  and modular on  $\mathfrak{Y}$  for  $x = x_0$  uniformly  $(\mathfrak{P}; \sigma)$ , then there exists a neighborhood  $(x_0)_\alpha$  of  $x_0$  such that  $F$  is modular on  $\mathfrak{Y}$  uniformly  $((x_0)_\alpha \mathfrak{P}; \sigma)$ .*

This is an immediate consequence of the definitions. More generally it is possible to show that if  $F$  is continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{Y}; \|\cdot\| \sigma)$  and modular on  $\mathfrak{Y}$  uniformly  $(\mathfrak{P}; \sigma)$  for every  $x_0$  of  $\mathfrak{X}_0$ , then the modulus of  $F$  on  $\mathfrak{Y}$  is continuous on  $\mathfrak{X}_0$ .

**11. The differential.** The function  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  is said to have a differential at  $x_0$  of  $\mathfrak{X}_0$  if there exists a function  $dF$  on  $\mathfrak{X}$  to  $\mathfrak{Y}$  linear on  $\mathfrak{X}$  such that the function  $R$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  defined by the conditions

$$F(x_1) - F(x_0) - dF(x_1 - x_0) = R(x_1) \|x_1 - x_0\| \text{ for } x_1 \neq x_0,$$

$$y_* = R(x_1) \text{ for } x_1 = x_0,$$

is continuous in  $x_1$  at  $x_0$ , i. e.,

$$\lim_{x_1 \rightarrow x_0} \|R(x_1)\| = 0.*$$

It is natural to denote the argument of the function  $dF$  by  $dx$ . The range of the variable  $dx$  is then always the whole space  $\mathfrak{X}$ . In the special case  $F(x) = x$ , we have  $dF(dx) = dx$ .

We have at once the following result:

**LEMMA 11.1.** *If  $F$  is on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  and has a differential at  $x_0$  of  $\mathfrak{X}_0$ , then this differential is unique and can be obtained as*

$$\lim_{\alpha \rightarrow 0} \frac{F(x_0 + \alpha dx) - F(x_0)}{\alpha} \quad \text{or} \quad \left. \frac{d}{d\alpha} F(x_0 + \alpha dx) \right|_{\alpha=0},$$

the limit being taken in the sense of norm.

This is an immediate consequence of propositions 1.39 and 2.12.

**12. The class  $\mathfrak{C}'$ .** As in ordinary analysis we get a class  $\mathfrak{C}'$  of functions  $F$  by limiting ourselves to functions  $F$  whose differentials  $dF$  have certain continuity properties on  $\mathfrak{X}_0$ . We define†

\*We follow here the Stolz-Young definition of differential, whose applicability to functional analysis was emphasized by Fréchet. For a discussion of different types of definitions of differentials in the case of functionals cf. Lévy, *Analyse Fonctionnelle*, pp. 50 ff.

†Cf. Bolza, *Variationsrechnung*, p. 13.

$F$  on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$  in case there exists a function  $dF$  on  $\mathfrak{X}_0\mathfrak{X}\mathfrak{P}$  to  $\mathfrak{Y}$  having the following properties:

- (1)  $dF$  is continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{X}\mathfrak{P}; \|dx\|\sigma)$ ;
- (2) for every  $x_0$  of  $\mathfrak{X}_0$ ,  $dF$  is linear uniformly  $(\mathfrak{P}; \sigma)$ ;
- (3) for every  $x_0$  of  $\mathfrak{X}_0$  the function  $R$  on  $\mathfrak{X}_0\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$  defined by

$$F(x_1) - F(x_0) - dF(x_0; x_1 - x_0) = R(x_1, x_0)\|x_1 - x_0\| \text{ for } x_1 \neq x_0,$$

$$y_* = R(x_1, x_0) \quad \text{for } x_1 = x_0,$$

is continuous in  $x_1$  at  $x_0$  uniformly  $(\mathfrak{P}; \sigma)$ .

Obviously this is not the only method of defining the concept "function  $F$  of class  $\mathfrak{C}'$ ", but it is the simplest direct generalization of the usual definition which adapts itself to elegant results. Another useful concept is obtained by the addition of the range  $\mathfrak{X}_0$  to the range of uniformity in the above conditions on  $dF$ . We shall call such a class of functions "class  $\mathfrak{C}'$  uniformly  $(\mathfrak{X}_0\mathfrak{P}; \sigma)$ ."\*

Most of the lemmas which we shall derive hold in two ways, obtained by adding or omitting the class  $\mathfrak{X}_0$  in the range of uniformity in hypothesis and conclusion. In most cases the proof of one lemma thus obtained can be derived from that of the other without difficulty.

**LEMMA 12.1.** *If  $F$  on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$  then  $F$  is continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$ . Moreover for every  $x_0$  of  $\mathfrak{X}_0$  there exists a vicinity  $(x_0)_a$  and an  $M$  such that for all points  $x$  of  $(x_0)_a$  and all  $p$  of  $\mathfrak{P}$  it is true that*

$$\|F(x) - F(x_0)\| \leq M\|x - x_0\| |\sigma|$$

and

$$\|dF(x, dx)\| \leq M\|dx\| |\sigma|,$$

i. e.  $F$  is linear on  $\mathfrak{X}$  uniformly  $((x_0)_a\mathfrak{P}; \sigma)$ .

The first of these is an immediate consequence of the modularity of  $dF$  and the application of condition (3) to the inequality

$$\|F(x) - F(x_0)\| \leq \|dF(x_0, x - x_0)\| + \|R(x, x_0)\| \|x - x_0\|.$$

A similar result (without the addition of the uniformity as to  $\sigma$ ) follows immediately from the existence of the differential.

The second result is an immediate consequence of Lemma 10.1.

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\*It is possible to show that if  $F$  is of class  $\mathfrak{C}'$  uniformly  $(\mathfrak{P}; \sigma)$  then the conditions (1) and (2) on  $dF$  hold uniformly  $(\mathfrak{X}_0\mathfrak{P}; \sigma)$  for every compact subclass  $\mathfrak{X}_{00}$  of  $\mathfrak{X}_0$ .

LEMMA 12.2. *If  $F$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{X}_0\mathfrak{P}; \sigma)$ , then  $F$  is continuous uniformly  $(\mathfrak{X}_0\mathfrak{P}; \sigma)$  and there exist constants  $a$  and  $M$  such that for every  $x_1$  and  $x_2$  of  $\mathfrak{X}_0$  for which*

$$\|x_1 - x_2\| \leq a$$

*and for every  $p$  of  $\mathfrak{P}$ , it is true that*

$$\|F(x_1) - F(x_2)\| \leq M\|x_1 - x_2\| |\sigma|,$$

*i. e.,  $F$  satisfies a kind of Lipschitz condition on  $\mathfrak{X}_0$ .\**

13. **Partial differentials.** If the class  $\mathfrak{X}$  is a composite of a finite number of classes  $\mathfrak{X}^{(1)}, \mathfrak{X}^{(2)}, \dots, \mathfrak{X}^{(n)}$ , and  $F$  is on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$ , then the existence of the differential of  $F$  for  $x = x_0$  will imply the existence of partial differentials and we shall have

$$dF(x_0; dx) = \sum_{i=1}^n d_{x^{(i)}}F(x_0; dx^{(i)}),$$

where  $dx = (dx^{(1)}, \dots, dx^{(n)})$ . In the same way if  $F$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  then the partial differentials have certain continuity properties. We state them in the following lemma for the case in which  $n=2$ .

LEMMA 13.1. *If  $\mathfrak{X}_0 = \mathfrak{X}'_0 \mathfrak{X}''_0$  and  $F$  on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$  then the partial differentials  $d_{x'}F$  and  $d_{x''}F$  have the following properties:*

- (0)  $dF(x; dx) = d_{x'}F(x; dx') + d_{x''}F(x; dx'')$ ;
- (1)  $d_{x'}F$  and  $d_{x''}F$  are continuous on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{X}'\mathfrak{P}; \|dx'\| \sigma)$  and  $(\mathfrak{X}''\mathfrak{P}; \|dx''\| \sigma)$  respectively;
- (2)  $d_{x'}F$  and  $d_{x''}F$  are linear in  $dx'$  and  $dx''$  respectively, uniformly  $(\mathfrak{P}; \sigma)$ ;
- (3) if  $R_{x'}$  on  $\mathfrak{X}_0\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$  is defined by

$$F(x'_1 x''_1) - F(x'_0 x''_0) - d_{x'}F(x'_0 x''_0; x'_1 - x'_0) = R_{x'}(x_1 x_0) \|x_1 - x_0\|$$

*for  $x_1 \neq x_0$ ,*

$$y_* = R_{x'}(x_1 x_0) \text{ for } x_1 = x_0,$$

\* A similar result for  $x_1$  and  $x_2$  in a neighborhood  $(x_0)_a$  of  $x_0$  ( $a$  depending on  $x_0$ ) could be deduced under the hypothesis of uniformity on  $\mathfrak{P}$  only, if condition (3) on the class  $\mathfrak{C}'$  were replaced by the condition

$$\lim_{\substack{x_1 \rightarrow x_0, x_2 \rightarrow x_0}} \|R(x_1, x_2)\| = 0 \text{ uniformly } (\mathfrak{P}; \sigma).$$

This extended result is however also deducible from the original condition (3) by an application of Taylor's theorem. Cf. L. M. Graves, *Riemann integration and Taylor's theorem in general analysis*, in the present number of these Transactions.

then  $R_{x'}(x_1, x_0)$  considered as a function of  $x_1$  is continuous at  $x_0$  uniformly ( $\mathfrak{P}; \sigma$ ). A similar condition holds for  $R_{x''}(x_1, x_0)$  similarly defined. Conversely, if  $d_{x'}F$  and  $d_{x''}F$  satisfy the conditions (1), (2), (3) on  $\mathfrak{X}_0$  uniformly ( $\mathfrak{P}; \sigma$ ) then  $F$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly ( $\mathfrak{P}; \sigma$ ) and  $dF$  can be defined by condition (0).

The properties (0), (1), and (2) are obvious. For (3) we use the existence of  $d_{x''}F$  and  $dF = d_{x'}F + d_{x''}F$  uniformly ( $\mathfrak{P}; \sigma$ ). The proof of the converse is obvious.

In case  $F$  is of class  $\mathfrak{C}'$  uniformly ( $\mathfrak{X}_0\mathfrak{P}; \sigma$ ) we have the following simpler lemma:

**LEMMA 13.2.** *If  $\mathfrak{X}_0 = \mathfrak{X}'_0 \mathfrak{X}''_0$  and  $F$  is on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Y}$ , then  $F$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly ( $\mathfrak{X}_0\mathfrak{P}; \sigma$ ) if and only if the partial differentials  $d_{x'}F$  and  $d_{x''}F$  have properties similar to those of  $dF$  uniformly ( $\mathfrak{X}_0\mathfrak{P}; \sigma$ ).*

These lemmas emphasize the fact that the partial differentials are functions on  $\mathfrak{X}_0\mathfrak{X}'$  and  $\mathfrak{X}_0\mathfrak{X}''$  respectively, and that the limit involved in the total differential is essentially a two-dimensional one.

**14. Higher differentials.** The class  $\mathfrak{C}^{(n)}$ . If for some neighborhood of  $x_0$ ,  $dF(x; d_1x)$  exists and if this function has for every  $d_1x$  a differential at  $x_0$ , then  $F$  is said to have a second differential,  $d^2F(x_0; d_1x, d_2x)$ . This function will be distributive in  $d_1x$  but not necessarily modular. If the function  $d^2F(x_0; d_1x, d_2x)$  is modular in  $d_2x$  uniformly ( $\mathfrak{X}; \|d_1x\|$ ) then obviously  $d^2F$  will be bilinear in  $d_1x$  and  $d_2x$ ; i. e.

$$\begin{aligned} d^2F(x_0; d'_1x + d''_1x, d'_2x + d''_2x) &= d^2F(x_0; d'_1x, d'_2x) \\ &+ d^2F(x_0; d'_1x, d''_2x) + d^2F(x_0; d''_1x, d'_2x) + d^2F(x_0; d''_1x, d''_2x); \end{aligned}$$

and there exists a constant  $M$  such that, for every  $d_1x$  and  $d_2x$  of  $\mathfrak{X}$ ,

$$\|d^2F(x_0; d_1x, d_2x)\| \leq M\|d_1x\| \cdot \|d_2x\|.$$

Without further assumptions on  $d^2F$  it does not seem possible to prove that  $d^2F$  is symmetric in  $d_1x$  and  $d_2x$ .

Obviously we can extend these definitions to the  $n$ th differential and we have, for every  $k < n$ ,

$$d^{n-k}d^kF = d^nF,$$

i. e. the differential operation is associative. But the successive differential operations are not necessarily commutative.

We define the term " $F$  is of class  $\mathfrak{G}^{(n)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{P}; \sigma)^{n*}$  by the following recurrence relation:

$F$  is of class  $\mathfrak{G}^{(n)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{P}; \sigma)$  if  $F$  is of class  $\mathfrak{G}'$  uniformly  $([\mathfrak{X}_0]\mathfrak{P}; \sigma)$  and  $dF$  is of class  $\mathfrak{G}^{(n-1)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{X}\mathfrak{P}; \|d_1x\|\sigma)$ .

It is possible to state equivalent definitions as indicated by the following

**LEMMA 14.1.** *If  $k$  is an integer between 0 and  $n$ , and if  $F$  is of class  $\mathfrak{G}^{(k)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{P}; \sigma)$  and  $d^kF$  is of class  $\mathfrak{G}^{(n-k)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^k \|d_ix\|\sigma)$ , then  $F$  is of class  $\mathfrak{G}^{(n)}$  uniformly  $([\mathfrak{X}_0]\mathfrak{P}; \sigma)$  and conversely.*

For convenience we shall omit the range  $[\mathfrak{X}_0]\mathfrak{P}$  and the scale  $\sigma$  of uniformity from the discussion. We proceed by induction and assume that we have proved the lemma (necessary and sufficient conditions) for all values of  $k \leq m$  and for all  $n > m$ . We show then

(a) if  $F$  is of class  $\mathfrak{G}^{(m+1)}$  and  $d^{m+1}F$  is of class  $\mathfrak{G}^{(n-(m+1))}$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^{m+1} \|d_ix\|)$ , then  $F$  is of class  $\mathfrak{G}^{(n)}$ .

For if  $F$  is of class  $\mathfrak{G}^{(m+1)}$  then  $F$  is of class  $\mathfrak{G}'$  and  $dF$  is of class  $\mathfrak{G}^{(m)}$  uniformly  $(\mathfrak{X}; \|d_1x\|)$ . On the other hand, the statement that  $d^{m+1}F$  is of class  $\mathfrak{G}^{(n-m-1)}$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^{m+1} \|d_ix\|)$  is equivalent to  $d^m(dF)$  is of class  $\mathfrak{G}^{(n-m-1)}$  in the same way. Applying the lemma for  $k=m$ , it follows that  $dF$  is of class  $\mathfrak{G}^{(n-m-1+m)}$  or  $\mathfrak{G}^{(n-1)}$  uniformly  $(\mathfrak{X}; \|d_1x\|)$ . This together with the fact that  $F$  is of class  $\mathfrak{G}'$  gives the result that  $F$  is of class  $\mathfrak{G}^{(n)}$ .

On the other hand

(b) if  $F$  is of class  $\mathfrak{G}^{(m)}$  and  $d^mF$  is of class  $\mathfrak{G}^{(n-m)}$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^m \|d_ix\|)$ , then  $F$  is of class  $\mathfrak{G}^{(m+1)}$  and  $d^{m+1}F$  is of class  $\mathfrak{G}^{(n-m-1)}$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{j=1}^{m+1} \|d_jx\|)$ , i. e. we can go from  $k=m$  to  $k=m+1$ .

For from the second condition it follows that  $d^mF$  is of class  $\mathfrak{G}'$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^m \|d_ix\|)$  and  $d^{m+1}F$  is of class  $\mathfrak{G}^{(n-m-1)}$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^{m+1} \|d_ix\|)$ . By applying the lemma for  $k=m$  and  $n=m+1$ , we find that if  $F$  is of class  $\mathfrak{G}^{(m)}$  and  $d^mF$  is of class  $\mathfrak{G}'$  uniformly  $(\mathfrak{X} \cdots \mathfrak{X}; \prod_{i=1}^m \|d_ix\|)$  then  $F$  is of class  $\mathfrak{G}^{(m+1)}$ . We thus get the two conditions of the conclusion from the two conditions of the hypothesis of our statement (b). This completes the proof of the lemma.

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\*By enclosing  $\mathfrak{X}_0$  in the bracket [ ] we shall indicate the possibility of its omission from the range of uniformity.

15. **Functions of functions.** LEMMA 15.1 *If  $G$  on  $\mathfrak{X}_0\mathfrak{Y}\mathfrak{Z}$  to  $\mathfrak{Z}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{X}_0$  uniformly ( $\mathfrak{Y}\mathfrak{Z}$ ;  $\|y\| \sigma$ ) and linear on  $\mathfrak{Y}$  for every  $x$  of  $\mathfrak{X}_0$  uniformly ( $\mathfrak{Z}$ ;  $\sigma$ ); if moreover  $H$  on  $\mathfrak{U}_0$  to  $\mathfrak{X}_0$  is of class  $\mathfrak{C}'$  on  $\mathfrak{U}_0$ , and  $K$  on  $\mathfrak{U}_0\mathfrak{V}$  to  $\mathfrak{Y}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{U}_0$  uniformly ( $\mathfrak{V}$ ;  $\|v\|$ ) and linear on  $\mathfrak{V}$  for each  $u$  of  $\mathfrak{U}_0$ , then*

$$G(u, v, p) = G(H(u), K(u, v), p)$$

*on  $\mathfrak{U}_0\mathfrak{V}\mathfrak{Z}$  to  $\mathfrak{Z}$  is of class  $\mathfrak{C}'$  on  $\mathfrak{U}_0$  uniformly ( $\mathfrak{V}\mathfrak{Z}$ ;  $\|v\| \sigma$ ) and linear on  $\mathfrak{V}$  for each  $u$  of  $\mathfrak{U}_0$  uniformly ( $\mathfrak{Z}$ ;  $\sigma$ ); moreover*

$$d_u G(u, v, p; du) = d_u G(H(u), K(u, v), p; dH(u; du)) + G(H(u), dK(u, v; du), p).$$

The fact that  $G(H(u), K(u, v), p)$  is linear on  $\mathfrak{V}$  for each  $u$  of  $\mathfrak{U}_0$  uniformly ( $\mathfrak{Z}$ ;  $\sigma$ ) is obvious. We show that  $G$  on  $\mathfrak{U}_0\mathfrak{V}\mathfrak{Z}$  to  $\mathfrak{Z}$  is of class  $\mathfrak{C}'$  uniformly ( $\mathfrak{V}\mathfrak{Z}$ ;  $\|v\| \sigma$ ) by showing that  $d_u G$  satisfies the conditions imposed in the definition of the class  $\mathfrak{C}'$ . For simplicity we shall omit the range  $\mathfrak{Z}$  and the scale function  $\sigma$  from the argument. It is easy to see that their addition does not change the form of the reasoning.

We shall show in the first place that  $d_u G$  is continuous on  $\mathfrak{U}_0$  uniformly ( $\mathfrak{U}\mathfrak{V}$ ;  $\|du\| \cdot \|v\|$ ). We consider for this purpose each term of  $d_u G$  separately and study the difference

$$\begin{aligned} & dG(H(u), K(u, v); dH(u; du)) - dG(H(u_0), K(u_0, v); dH(u_0; du)) \\ &= [dG(H(u), K(u, v); dH(u; du)) - dG(H(u_0), K(u, v); dH(u; du))] \\ &\quad + dG(H(u_0), K(u, v); dH(u; du) - dH(u_0; du)) \\ &\quad + dG(H(u_0), K(u, v) - K(u_0, v); dH(u_0; du)). \end{aligned}$$

In order to show that each of these three terms approaches  $z_*$ , we assume that  $u$  has been chosen in the vicinity of  $u_0$  on which, by Lemmas 10.1 and 12.1,  $dH(u; du)$  is uniformly modular in  $du$ , and  $K(u, v)$  uniformly modular in  $v$ . Then the continuity of  $dG$  uniformly relative to

$$\|dH(u; du)\| \cdot \|K(u, v)\|$$

and the continuity of  $H$  suffice to make the first difference approach  $z_*$  uniformly ( $\mathfrak{U}\mathfrak{V}$ ;  $\|du\| \cdot \|v\|$ ); the modularity of  $dG$  uniformly relative to  $\|K(u, v)\|$  and the continuity of  $dH(u; du)$  uniformly relative to  $\|du\|$  produce the approach to  $z_*$  uniformly ( $\mathfrak{U}\mathfrak{V}$ ;  $\|du\| \cdot \|v\|$ ) for the second term, while the additional continuity of  $K(u, v)$  uniformly ( $\mathfrak{V}$ ;  $\|v\|$ ) produces the same result in the third term.

The continuity of  $G(H(u); dK(u, v; du))$  is proved along entirely similar lines.

The modularity of  $d_u G$  follows immediately from that of  $d_x G$ ,  $G$ ,  $dH$  and  $dK$ .

We consider finally

$$\begin{aligned} R(u_1, u_0) \|u_1 - u_0\| &= G(H(u_1), K(u_1, v)) - G(H(u_0), K(u_0, v)) \\ &\quad - d_x G(H(u_0), K(u_0, v); dH(u_0; u_1 - u_0)) \\ &\quad - G(H(u_0), dK(u_0, v; u_1 - u_0)) \\ &= [G(H(u_1), K(u_1, v) - K(u_0, v)) \\ &\quad - G(H(u_0), dK(u_0, v; u_1 - u_0))] \\ &\quad + [G(H(u_1), K(u_0, v)) - G(H(u_0), K(u_0, v)) \\ &\quad - dG(H(u_0), K(u_0, v); dH(u_0; u_1 - u_0))]. \end{aligned}$$

The first group of terms we rewrite in the form

$$\begin{aligned} &[G(H(u_1), K(u_1, v) - K(u_0, v) - dK(u_0, v; u_1 - u_0))] \\ &\quad + [G(H(u_1), dK(u_0, v; u_1 - u_0)) - G(H(u_0), dK(u_0, v; u_1 - u_0))]. \end{aligned}$$

If now we assume that  $u_1$  is in such a neighborhood of  $u_0$  that  $H(u_1)$  lies in the neighborhood of  $H(u_0)$  for which, by Lemma 10.1,  $G$  is uniformly modular on  $\mathcal{Y}$ , it follows that the norm of the first expression is less than or equal to

$$M_G \|R_K(u_1, u_0)\| \cdot \|u_1 - u_0\|$$

in which the coefficient of  $\|u_1 - u_0\|$  approaches zero uniformly ( $\mathcal{B}; \|v\|$ ) as  $u_1$  approaches  $u_0$ . For the second expression we utilize the continuity of  $G$  uniformly ( $\mathcal{Y}; \|y\|$ ), i. e. ( $\mathcal{Y}; \|dK(u_0, v; u_1 - u_0)\|$ ) which in turn can

be replaced by uniformly  $(\mathfrak{Y}; M_k \|v\| \cdot \|u - u_0\|)$  so that the second term when divided by  $\|u_1 - u_0\|$  will approach  $z_*$  uniformly  $(\mathfrak{B}; \|v\|)$  as  $u_1$  approaches  $u_0$ .

The second group of terms in the expression for  $R(u_1, u_0)\|u_1 - u_0\|$  can be replaced by

$$R_G(H(u_1), H(u_0); K(u_0, v))\|H(u_1) - H(u_0)\| \\ + dG(H(u_0), K(u_0, v); R_H(u_1, u_0)\|u_1 - u_0\|).$$

By Lemma 12.1 if  $u_1$  is in a sufficiently small neighborhood of  $u_0$  there exists an  $M$  such that

$$\|H(u_1) - H(u_0)\| \leq M\|u_1 - u_0\|.$$

Hence the first term when divided by  $\|u_1 - u_0\|$  will approach  $z_*$  uniformly  $(V; \|v\|)$ . The same result for the second term follows from the modular properties of  $dG$ . This completes the proof of

$$\lim_{u_1 \rightarrow u_0} \|R(u_1, u_0)\| = 0 \text{ uniformly } (\mathfrak{B}; \|v\|).$$

We note that the addition of further linearity in the function  $K$  will induce corresponding linearity in  $G$ , i. e., if  $K(u; v_1, \dots, v_n)$  on  $\mathfrak{U}_0 \mathfrak{B}_1 \cdots \mathfrak{B}_n$  is of class  $\mathfrak{C}'$  on  $\mathfrak{U}_0$  uniformly  $(\mathfrak{B}_1 \cdots \mathfrak{B}_n; \prod_{i=1}^n \|v_i\|)$  and linear on each  $\mathfrak{B}_i$  uniformly  $(\mathfrak{B}_1 \cdots \mathfrak{B}_{i-1} \mathfrak{B}_{i+1} \cdots \mathfrak{B}_n; \prod_{i \neq j} \|v_j\|)$  then  $G$  has the same properties as  $K$ .

It is possible to extend this lemma still further. To simplify the statement we introduce the following terminology:  $G$  on  $\mathfrak{X}_0 \mathfrak{Y}_1 \cdots \mathfrak{Y}_m \mathfrak{P}$  has the property  $P^{(n)}$  on  $\mathfrak{X}_0 \mathfrak{Y}_1 \cdots \mathfrak{Y}_m$  uniformly  $(\mathfrak{P}; \sigma)$ , in case  $G$  is of class  $\mathfrak{C}^{(n)}$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{Y}_1 \cdots \mathfrak{Y}_m \mathfrak{P}; \prod_{i=1}^m \|y_i\| \sigma)$  and for each  $x$  of  $\mathfrak{X}_0$ ,  $G$  is linear on  $\mathfrak{Y}_i$  uniformly  $(\mathfrak{Y}_1 \cdots \mathfrak{Y}_{i-1} \mathfrak{Y}_{i+1} \cdots \mathfrak{Y}_m \mathfrak{P}; \prod_{i \neq j} \|y_j\| \sigma)$ .

Obviously if  $G$  has the property  $P'$  and  $dG$  has the property  $P^{(n-1)}$  uniformly  $(\mathfrak{X}; \|dx\|)$  then  $G$  has the property  $P^{(n)}$ .

We then have the following lemma:

LEMMA 15.2. If  $G$  on  $\mathfrak{X}_0 \mathfrak{Y}_1 \cdots \mathfrak{Y}_k \mathfrak{P}$  to  $\mathfrak{Z}$  has the property  $P^{(n)}$  on  $\mathfrak{X}_0 \mathfrak{Y}_1 \cdots \mathfrak{Y}_k$  uniformly  $(\mathfrak{P}; \sigma)$ ; if, moreover,  $H$  on  $\mathfrak{U}_0$  to  $\mathfrak{X}_0$  is of class  $\mathfrak{C}^{(n)}$  on  $\mathfrak{U}_0$ , and  $K_i$  on  $\mathfrak{U}_0 \mathfrak{B}_{i1} \cdots \mathfrak{B}_{ij_i}$  to  $\mathfrak{Y}_i$  has the property  $P^{(n)}$  on  $\mathfrak{U}_0 \mathfrak{B}_{i1} \cdots \mathfrak{B}_{ij_i}$ , then  $G$  on  $\mathfrak{U}_0 \mathfrak{B}_{11} \cdots \mathfrak{B}_{kj_k} \mathfrak{P}$  has the property  $P^{(n)}$  on  $\mathfrak{U}_0 \mathfrak{B}_{11} \cdots \mathfrak{B}_{kj_k}$  uniformly  $(\mathfrak{P}; \sigma)$ .

The proof for  $n=1$  and  $k=1$  has been given in Lemma 15.1 and the appended remark. Assume that it has been proved for values  $n=1$  and

$k \leq m$  and show that it holds for  $k = m + 1$ . This is a result of applying the lemma for  $k = m$  and  $k = 1$  to the following functions:

$$\begin{aligned}\bar{G}(u, y_{m+1}, v_{11}, \dots, v_{mj_m}, p) &= G(H, K_1, \dots, K_m, y_{m+1}, p), \\ \bar{H}(u) &= u, \\ \bar{K}(u, v_{m+1,1}, \dots, v_{m+1,j_{m+1}}) &= y_{m+1} = K_{m+1}(u, v_{m+1,1}, \dots, v_{m+1,j_{m+1}}).\end{aligned}$$

To prove the lemma for any  $n$ , assume that it has been proved for  $n \leq m$  and extend it to the case  $n = m + 1$ . It is obviously sufficient to show that  $d_u G$  has the property  $P^{(m)}$  uniformly  $(\mathfrak{P}; \sigma)$ . Now

$$\begin{aligned}d_u G &= d_x G(H, K_1, \dots, K_k, p; dH(du)) \\ &+ \sum_{i=1}^k G(H, K_1, \dots, dK_i(du), K_k, p).\end{aligned}$$

Obviously  $H$  is of class  $\mathfrak{C}^{(m)}$ , and  $d_x G, dH, K_1, \dots, K_k, dK_1, \dots, dK_k$  have the property  $P^{(m)}$  in their respective arguments. From the relation of  $P^{(m)}$  to the class  $\mathfrak{C}^{(m)}$  it follows that  $d_u G$  has the property  $P^{(m)}$  uniformly  $(\mathfrak{U}\mathfrak{P}; \|du\|\sigma)$  and consequently  $G$  has the property  $P^{(m+1)}$  uniformly  $(\mathfrak{P}; \sigma)$ .

As a special case of this lemma we note the following

**LEMMA 15.3.** *If  $G$  is on  $\mathfrak{X}_0\mathfrak{P}$  to  $\mathfrak{Z}$  of class  $\mathfrak{C}^{(n)}$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{P}; \sigma)$  and  $H$  is on  $\mathfrak{U}_0$  to  $\mathfrak{X}_0$  of class  $\mathfrak{C}^{(n)}$  on  $\mathfrak{U}_0$ , then  $G(H(u), p)$  on  $\mathfrak{U}_0\mathfrak{P}$  to  $\mathfrak{Z}$  is of class  $\mathfrak{C}^{(n)}$  on  $\mathfrak{U}_0$  uniformly  $(\mathfrak{P}; \sigma)$ .*

We need only apply the preceding lemma to the following situation:

$\mathfrak{P} = \mathfrak{Y} = \mathfrak{R}$  = the class of real numbers,

$K(u, r) = r$  for every  $u$  of  $\mathfrak{U}_0$ ,

$G(x, r) = rG(x)$  and  $H(u) = H(u)$ .

We note that if  $G$  is of class  $\mathfrak{C}$  and  $H$  of class  $\mathfrak{C}'$  then

$$d_u G(H(u)) = d_x G(H(u); dH(u, du)),$$

a function of the type considered in Lemma 15.1.

In Lemmas 15.1, 15.2 and 15.3 it is possible to get a parallel group of lemmas by inserting the classes  $\mathfrak{X}_0$  and  $\mathfrak{U}_0$  in the range of uniformity. In this case, especially for Lemma 15.1, the proof is simplified in a few places. As a matter of fact it is possible in this case to prove Lemma 15.3 directly by a line of reasoning which considers  $dG$  as a function of class  $\mathfrak{C}^{(m)}$  of a two-partite argument, each argument in turn being a function on a two-partite class.

## IV. ON THE RECIPROCAL OF LINEAR FUNCTIONS

16. In this section we shall assume that  $\mathfrak{Y}$  is a complete linear metric space and  $\mathfrak{X}$  is a linear metric space.

We say that a function  $K$  on  $\mathfrak{Y}$  to  $\mathfrak{Y}$ , linear on  $\mathfrak{Y}$ , has a *reciprocal* if there exists a function  $L$  on  $\mathfrak{Y}$  to  $\mathfrak{Y}$  linear on  $\mathfrak{Y}$  satisfying the conditions

$$K(L(y)) = L(K(y)) = y$$

for every  $y$  of  $\mathfrak{Y}$ .

We note that if such an  $L$  exists, then it is unique. For from

$$K(L_1(y)) = K(L_2(y)) = y$$

follows

$$L_2KL_1 = L_2KL_2 \text{ or } L_1(y) = L_2(y).$$

Also from the existence of the reciprocal  $L$  it follows that if

$$L(y) = y_*$$

for a given  $y$ , then  $y = y_*$ . Conversely if there exists an  $L$  on  $\mathfrak{Y}$  to  $\mathfrak{Y}$  linear on  $\mathfrak{Y}$  satisfying the two conditions

$$L(K(y)) = y \text{ for every } y \text{ of } \mathfrak{Y},$$

$$L(y) = y_* \text{ if and only if } y = y_*,$$

then

$$K(L(y)) = y \text{ for every } y \text{ of } \mathfrak{Y}.$$

For

$$L(KLy - y) = LKLy - Ly = Ly - Ly = y_*.$$

We take up first the reciprocal of a function which is suggested by the linear integral equation of the second kind in the following

LEMMA 16.1. *If  $G$  on  $\mathfrak{X}_0\mathfrak{Y}$  to  $\mathfrak{Y}$  is linear on  $\mathfrak{Y}$  for each  $w$  of  $\mathfrak{X}_0$  with modulus  $M(G, w) < 1$ , then for every  $w$  of  $\mathfrak{X}_0$  there exists a reciprocal of  $y - G(w, y)$ . If  $G$  is continuous on  $\mathfrak{X}_0$  uniformly ( $\mathfrak{Y}; \|y\|$ ) then the reciprocal is also continuous uniformly ( $\mathfrak{Y}; \|y\|$ ).*

The proof of the first part of this lemma follows the lines of the successive substitution method used in Theorem 1, or the lemma can be shown to be a corollary of Theorem 1, which amounts essentially to the same thing. From either point of view, it appears that if  $G_n(w, y) = G(w, G_{n-1}(w, y))$ ;  $G_1(w, y) = G(w, y)$ , then for all  $(w, y)$  of  $\mathfrak{X}_0\mathfrak{Y}$  the reciprocal of  $y - G(w, y)$  is

$$y - H(w, y) = y + \sum_1^{\infty} G_n(w, y),$$

and the modulus

$$M(H, w) \leq \frac{M(G, w)}{1 - M(G, w)}$$

Obviously  $H$  satisfies the relations

$$H(w, y) + G(w, y) = G(w, H(w, y)) = H(w, G(w, y))$$

which are reminiscent of the reciprocal relations of linear integral equations.

To show that the continuity of  $G$  on  $\mathfrak{B}_0$  uniformly  $(\mathfrak{Y}; \|y\|)$  implies a similar continuity in the function  $H$  we utilize the first of these reciprocal relations to obtain

$$\begin{aligned} H(w_1, y) + G(w_1, y) - G(w_1, H(w_1, y)) \\ = H(w_0, y) + G(w_0, y) - G(w_0, H(w_0, y)) \end{aligned}$$

from which follows

$$\begin{aligned} \|H(w_1, y) - H(w_0, y)\| &\leq \|G(w_1, y) - G(w_0, y)\| + \|G(w_1, H(w_1, y)) \\ &\quad - H(w_0, y)\| + \|G(w_1, H(w_0, y)) - G(w_0, H(w_0, y))\|. \end{aligned}$$

Now from the continuity of  $G(w, y)$  on  $\mathfrak{B}_0$  uniformly  $(\mathfrak{Y}; \|y\|)$  it follows that there will exist a vicinity  $(w_0)_a$  of  $w_0$  such that if  $w_1$  is in this vicinity then  $M(G, w_1) < k < 1$ , so that for such  $w_1$

$$\begin{aligned} \|H(w_1, y) - H(w_0, y)\|(1 - k) &\leq \|G(w_1, y) - G(w_0, y)\| \\ &\quad + \|G(w_1, H(w_0, y)) - G(w_0, H(w_0, y))\| \end{aligned}$$

The continuity properties of  $G$  applied to this inequality give us corresponding continuity properties of  $H$ .

By the same method we can obviously show that if  $G$  is continuous on  $\mathfrak{B}_0$  uniformly  $(\mathfrak{B}_0\mathfrak{Y}; \|y\|)$  then for every  $w_0$  there exists a vicinity  $(w_0)_a$  such that  $H$  is continuous on  $(w_0)_a$  uniformly  $((w_0)_a\mathfrak{Y}; \|y\|)$ . This vicinity can be extended to include the region  $\mathfrak{B}_0$  if there exists a  $k < 1$  such that for all  $w$  of  $\mathfrak{B}_0$

$$M(G, w) < k.$$

**LEMMA 16.2.** Suppose that  $K$  on  $\mathfrak{B}_0\mathfrak{Y}$  to  $\mathfrak{Y}$  is linear on  $\mathfrak{Y}$  for each  $w$  of  $\mathfrak{B}_0$ ; that it has a reciprocal  $L_0$  for  $w=w_0$ , and is continuous in  $w$  at  $w=w_0$  uniformly  $(\mathfrak{Y}; \|y\|)$ ; then there exists a constant  $a$  such that for each  $w$  of the neighborhood  $(w_0)_a$ ,  $K$  has a reciprocal, which is continuous at  $w_0$  uniformly  $(\mathfrak{Y}; \|y\|)$ .

Consider the function

$$G(w, y) = L_0(K(w_0, y) - K(w, y)) = y - L_0K(w, y).$$

Using the continuity of  $K$  at  $w_0$ , we can select a  $k$  and an  $a$  such that for all  $w$  in  $(w_0)_a$  we have

$$\|K(w_0, y) - K(w, y)\| < k\|y\|, \quad kM(L_0) < 1.$$

Then

$$M(G, w) < k_1 < 1$$

and so by Lemma 16.1,

$$L_0K(w, y) = y - G(w, y)$$

has on  $(w_0)_a$  a reciprocal  $L_1$  continuous at  $w=w_0$  uniformly  $(\mathfrak{Y}; \|y\|)$ . Then for every  $y$  of  $\mathfrak{Y}$  and  $w$  of  $(w_0)_a$ ,

$$L_1L_0Ky = L_0KL_1y = y,$$

where for convenience we have omitted the  $w$ . Then  $L_1L_0$  is a left hand reciprocal of  $K$ . That it is also a right hand reciprocal follows from

$$L_0KL_1L_0y = L_0y \quad \text{or} \quad L_0(KL_1L_0y - y) = y_*$$

and the property of  $L_0$  as reciprocal of  $K(w_0, y)$ . The continuity of  $L_1L_0$  at  $w=w_0$  uniformly  $(\mathfrak{Y}; \|y\|)$  follows from that of  $L_1$ .

If  $K$  is also continuous on  $\mathfrak{B}_0$  uniformly  $(\{\mathfrak{B}_0\}\mathfrak{Y}; \|y\|)$  then this reciprocal will be continuous on  $(w_0)_a$  uniformly  $(\{(w_0)_a\}\mathfrak{Y}; \|y\|)$ .

Differential properties of  $K$  carry over to the reciprocal as we show in

**LEMMA 16.3.** *If  $K$  on  $\mathfrak{B}_0\mathfrak{Y}$  to  $\mathfrak{Y}$  is linear on  $\mathfrak{Y}$  for each  $w$  of  $\mathfrak{B}_0$ , has a reciprocal  $L_0$  for  $w=w_0$  and is of class  $C^{(n)}$  on  $\mathfrak{B}_0$  uniformly  $(\{\mathfrak{B}_0\}\mathfrak{Y}; \|y\|)$ , then there exists a constant  $a$  such that the reciprocal  $L$  is of class  $C^{(n)}$  on  $(w_0)_a$  uniformly  $(\{(w_0)_a\}\mathfrak{Y}; \|y\|)$ .*

We select the constant  $a$  in such a way that the function  $L(w, y)$  is uniformly modular on  $(w_0)_a$ , which is always possible as can be seen from Lemma 10.1. We then prove this lemma first for the case  $n=1$  by showing that

$$(16.4) \quad dL(w, y; dw) = -L(w, dK(w, L(w, y); dw))$$

satisfies the three conditions imposed on differentials of functions of the class  $\mathfrak{C}'$ . The relation (16.4) can be obtained by applying Lemma 15.1 heuristically to the reciprocal relation

$$K(w, L(w, y)) = y.$$

The continuity of  $dL$  can be shown by following the method used, or the result obtained in the proof of the corresponding part of Lemma 15.1. Similarly the modularity of  $dL$  is deducible from that of  $L$  and  $dK$ . It remains to show that  $dL$  satisfies condition (3). From the reciprocal relation between  $K$  and  $L$  it follows that for  $w_1$  and  $w_2$  in  $(w_0)_a$  it is true that

$$K(w_1, L(w_1, y)) = K(w_2, L(w_2, y))$$

or

$$\begin{aligned} K(w_1, L(w_2, y)) - K(w_1, L(w_1, y)) &= - (K(w_2, L(w_2, y)) - K(w_1, L(w_2, y))) \\ &= - dK(w_1, L(w_2, y); w_2 - w_1) \\ &\quad - R_K(w_2, w_1, L(w_2, y)) \|w_2 - w_1\|. \end{aligned}$$

Then

$$\begin{aligned} L(w_2, y) - L(w_1, y) &= - L(w_1, dK(w_1, L(w_2, y); w_2 - w_1)) \\ &\quad - L(w_1, R_K(w_2, w_1, L(w_2, y))) \|w_2 - w_1\|, \end{aligned}$$

so that

$$\begin{aligned} R_L(w_2, w_1, y) \|w_2 - w_1\| &= L(w_2, y) - L(w_1, y) - dL(w_1, y; w_2 - w_1) \\ &= - L(w_1, dK(w_1, L(w_2, y) - L(w_1, y); w_2 - w_1)) \\ &\quad - L(w_1, R_K(w_2, w_1, L(w_2, y))) \|w_1 - w_1\|. \end{aligned}$$

Now by Lemma 12.1,  $K$  is continuous at  $w_1$  uniformly  $(\mathfrak{Y}; \|y\|)$  and so by Lemma 16.2  $L$  is continuous at  $w_1$  uniformly  $(\mathfrak{Y}; \|y\|)$ . Further, from the modularity of  $dK$  and  $L$  it follows that

$$\begin{aligned} \|L(w_1, dK(w_1, L(w_2, y) - L(w_1, y); w_2 - w_1))\| \\ \leq M(L)M(dK)\|L(w_2, y) - L(w_1, y)\| \|w_2 - w_1\|, \end{aligned}$$

so that the coefficient of  $\|w_2 - w_1\|$  approaches zero uniformly  $(\mathfrak{Y}; \|y\|)$  as  $w_2$  approaches  $w_1$ . Similarly, from the uniform modularity of  $L$  on  $(w_0)_a$  and the manner of approach of  $R_K$  to  $y_*$  it follows that

$$\|L(w_1, R_K(w_2, w_1, L(w_1, y)))\|$$

approaches zero uniformly  $(\mathfrak{Y}; \|y\|)$ . This completes the proof that  $R_L$  approaches  $y_*$  uniformly  $(\mathfrak{Y}; \|y\|)$  as  $w_2$  approaches  $w_1$ .

We prove the general case by induction, i. e. assume that the lemma holds for  $n \leq m$  and show its validity for  $n = m + 1$ . Assume then that  $K$  is of class  $\mathfrak{C}^{(m+1)}$  uniformly  $(\mathfrak{Y}; \|y\|)$ . Then  $dK$  is of class  $\mathfrak{C}^{(m)}$  on  $\mathfrak{B}_0$  uniformly  $(\mathfrak{Y}\mathfrak{B}; \|y\| \cdot \|dw\|)$  and also  $L$  is of class  $\mathfrak{C}^{(m)}$  on  $\mathfrak{B}_0$  uniformly  $(\mathfrak{Y}; \|y\|)$ .

Then by applying Lemma 15.2 we get that  $dK(w, L(w, y); dw)$  is of class  $\mathfrak{C}^{(m)}$  on  $\mathfrak{X}_0$  uniformly  $(\mathfrak{Y}\mathfrak{B}; \|y\| \cdot \|dw\|)$  and so by a second application of Lemma 15.2 it follows that  $L(w, dK(w, L(w, y); dw)) = -dL(w, y; dw)$  has the same property. This completes the proof of the lemma.

V. EXISTENCE AND DIFFERENTIABILITY OF SOLUTIONS OF  
EQUATIONS OF THE FORM  $G(x, y) = y_*$

17. Neighborhood theorems. THEOREM 3. Let  $\mathfrak{X}$  be a metric space and  $\mathfrak{Y}$  a complete linear metric space. Let the point  $y_0$  of  $\mathfrak{Y}$  and the region  $\mathfrak{X}_0$  of  $\mathfrak{X}$  and the function  $G$  on  $\mathfrak{X}_0(y_0)_a$  to  $\mathfrak{Y}$  be such that

(H<sub>1</sub>) there exist a linear function  $K$  on  $\mathfrak{Y}$  to  $\mathfrak{Y}$  with a reciprocal  $L$ , and a positive constant  $M_0$  such that, if  $M(L)$  denote the modulus of  $L$ , then

$$M_0 M(L) < 1, \|K(y_1 - y_2) - G(x, y_1) + G(x, y_2)\| \leq M_0 \|y_1 - y_2\|$$

for every  $x$  in  $\mathfrak{X}_0$  and every  $y_1$  and  $y_2$  in  $(y_0)_a$ ;

(H<sub>2</sub>) there exists a positive constant  $c < a$  such that

$$M(L) \|G(x, y_0)\| \leq (1 - M_0 M(L))c$$

for every  $x$  in  $\mathfrak{X}_0$ .

Then for every  $x$  in  $\mathfrak{X}_0$  there exists a unique solution  $Y(x)$  of the equation

$$(17.1) \quad G(x, y) = y_*.$$

Note that in Theorem 3, only an approximate initial solution, and also only an approximate differential at the approximate initial solution are required. The proof is as follows:

We define a new function  $F$  on  $\mathfrak{X}_0(y_0)_a$  to  $\mathfrak{Y}$  by the equality

$$F(x, y) = y - L(G(x, y)).$$

Then the equation

$$(17.2) \quad y = F(x, y)$$

is equivalent to (17.1). To equation (17.2) we can apply Theorem 1. For by  $H_1$  we have

$$F(x, y_1) - F(x, y_2) = L(K(y_1 - y_2) - G(x, y_1) + G(x, y_2)),$$

$$\|F(x, y_1) - F(x, y_2)\| \leq M(L) M_0 \|y_1 - y_2\|,$$

so that the function  $F$  satisfies  $H_1$  of Theorem 1. That the hypothesis  $H_2$  of Theorem 1 also holds for the function  $F$  is readily verified.

THEOREM 4. Let  $\mathfrak{X}$  be a linear metric space, and  $\mathfrak{Y}$  a complete linear metric space, and let  $\mathfrak{W}_0$  be a region of the composite space  $\mathfrak{W} = (\mathfrak{X}, \mathfrak{Y})$ . Let the function  $G$  on  $\mathfrak{W}_0$  to  $\mathfrak{Y}$  and the point  $(x_0, y_0)$  of  $\mathfrak{W}_0$  satisfy

$$(H_1) \quad G(x_0, y_0) = y_*$$

$$(H_2) \quad G \text{ is of the class } \mathfrak{C}^{(n)} \text{ on } \mathfrak{W}_0 [\text{uniformly on } \mathfrak{W}_0];$$

$$(H_3) \quad d_y G(x_0, y_0; dy) \equiv K_0(dy) \text{ has a reciprocal } L_0.$$

Then there exist positive constants  $a$  and  $b$  and a function  $Y$  on  $(x_0)_b$  to  $(y_0)_a$  with the following properties:

$$(C_1) \quad \text{the region } ((x_0)_b, (y_0)_a) \text{ is contained in } \mathfrak{W}_0;$$

$$(C_2) \quad \text{the point } (x, Y(x)) \text{ is a solution of the equation}$$

$$(17.3) \quad G(x, y) = y_*$$

for every  $x$  in  $(x_0)_b$ , and there is no other solution with the same  $x$ , having  $y$  in  $(y_0)_a$ ;

$$(C_3) \quad \text{the differential } d_y G(x, Y(x); dy) \text{ has a reciprocal for every } x \text{ in } (x_0)_b;$$

$$(C_4) \quad \text{the function } Y \text{ is of the class } \mathfrak{C}^{(n)} \text{ on } (x_0)_b [\text{uniformly on } (x_0)_b].$$

The parts in brackets in  $H_2$  and  $C_4$  constitute an alternative reading. We shall prove only the theorem with the parts in brackets omitted. The proof of the other theorem is parallel, except that no recourse to Taylor's theorem is required. We proceed as follows:

Let  $M(L_0)$  be the modulus of the reciprocal  $L_0$  of  $K_0$ , and let  $M_0$  be a constant satisfying the condition  $0 < M_0 M(L_0) < 1$ . By Taylor's theorem\* we have, for every  $(x, y_1), (x, y_2)$  in a sufficiently restricted neighborhood  $((x_0, y_0))_a$ ,

$$G(x, y_1) - G(x, y_2) = \int_0^1 d_y G(x, y_2 + (y_1 - y_2)r; y_1 - y_2) dr.$$

If we take  $a$  sufficiently small, the inequality

$$\|G(x, y_1) - G(x, y_2) - K_0(y_1 - y_2)\| \leq M_0 \|y_1 - y_2\|$$

follows from the continuity of  $d_y G$  at  $(x_0, y_0)$  and the properties of Riemann integrals.† Therefore the hypothesis  $H_1$  of Theorem 3 is fulfilled on this neighborhood. By Lemma 12.1 there exists a positive constant  $b \leq a$  such that  $H_2$  of Theorem 3 is fulfilled on  $(x_0)_b$ . Hence the conclusions  $C_1$  and  $C_2$  follow from Theorem 3.

\*See Graves, *Riemann integration and Taylor's theorem in general analysis*, already cited.

†Graves, loc. cit.

By Lemma 12.1 and Theorem 2, we find that the solution  $Y(x)$  is continuous on  $(x_0)_b$ . Then by Lemma 16.2, if the constant  $b$  is sufficiently restricted,  $d_y G(x, Y(x); dy)$  has a reciprocal  $L(x; dy)$  which is continuous on  $(x_0)_b$  uniformly  $(\mathfrak{Y}; \|dy\|)$  and linear on  $\mathfrak{Y}$  uniformly on  $(x_0)_b$ . This gives us  $C_3$ .

In proving  $C_4$  for  $n=1$ , we show that the differential  $dY$  is given by the formula

$$dY(x; dx) = -L(x; d_x G(x, Y(x); dx)).$$

The continuity and linearity of  $dY$  follow readily from the corresponding properties of  $L$  and  $d_x G$ . Since  $G$  is of class  $\mathfrak{G}'$  and  $G(x, Y) = y_*$ , we have

$$y_* = d_y G(x, Y; Y_1 - Y) + d_x G(x, Y; x_1 - x) + R_G(W_1, W) \|W_1 - W\|.$$

By applying the operation  $L$  to both sides of this equation, we obtain

$$(17.4) \quad y_* = Y_1 - Y - dY(x; x_1 - x) + L(x; R_G(W_1, W)) \|W_1 - W\|.$$

Here we have set  $W = (x, Y(x))$ , etc. If the ratio  $r = \|W_1 - W\| \div \|x_1 - x\|$  is bounded when  $x_1$  is in a sufficiently small neighborhood of  $x$ , then the remainder  $R_Y(x_1, x) = -L(x; R_G(W_1, W))r$  is continuous at  $x_1 = x$ , since we have already shown that the function  $L$  is linear and the function  $Y$  is continuous. To show that  $r$  is bounded, we evidently need consider only the points  $x_1$  at which  $\|W_1 - W\| = \|Y_1 - Y\|$ . At these points we have from equation (17.4),

$$\|Y_1 - Y\| \leq M(dY) \|x_1 - x\| + M(L) \|R_G\| \|Y_1 - Y\|.$$

Select a constant  $c$  so small that whenever  $x_1$  is in  $(x)$ , we have  $M(L) \|R_G\|$  less than a constant  $e < 1$ . Then by transposing we obtain  $r(1-e) \leq M(dY)$ .

To complete the proof of  $C_4$  for all values of  $n$ , we assume it is true for  $n=m$  and that  $G$  is of class  $\mathfrak{G}^{(m+1)}$  on  $\mathfrak{B}_0$ . Then by Lemma 15.3,  $d_y G(x, Y; dy)$  and  $d_x G(x, Y; dx)$  are of class  $\mathfrak{G}^{(m)}$  on  $(x_0)_b$  uniformly  $(\mathfrak{Y}; \|dy\|)$  and  $(\mathfrak{X}; \|dx\|)$ , respectively. Next, by Lemma 16.3, the reciprocal of  $d_y G(x, Y; dy)$  or  $L(x; dy)$  is of class  $\mathfrak{G}^{(m)}$  on  $(x_0)_b$  uniformly  $(\mathfrak{Y}; \|dy\|)$ . Then by Lemma 15.2,  $dY$  is of class  $\mathfrak{G}^{(m)}$  on  $(x_0)_b$  uniformly  $(\mathfrak{X}; \|dx\|)$ , so that, by definition,  $Y$  is of class  $\mathfrak{G}^{(m+1)}$  on  $(x_0)_b$ .

18. The unique maximal sheet of solutions containing a given solution. In Theorem 4, the existence of a solution was shown only in a restricted neighborhood of the given solution. However, the region on which the solution is defined can frequently be extended beyond such a neighborhood, by a kind of process of continuation. The maximal sheet of solutions through

a given solution, as defined below, is limited only by boundary points of the region  $\mathfrak{B}_0$  and by points at which the differential  $d_y G(x, y; dy)$  has no reciprocal. The theorem is a generalization of one given by Bliss in his *Princeton Colloquium Lectures* (p. 22), and it will be convenient to use a slight modification of his terminology here.

A *sheet of points* in the space  $\mathfrak{B} = (\mathfrak{X}, \mathfrak{Y})$  is a set  $\mathfrak{B}^{(0)}$  of points  $w = (x, y)$  of  $\mathfrak{B}$  with the following properties:

(a) for every  $w_0$  of  $\mathfrak{B}^{(0)}$  there exist positive constants  $a$  and  $b \leq a$  such that, no two points of  $\mathfrak{B}^{(0)}$  in  $(w_0)_a$  have the same "projection"  $x$ , and every point  $x$  in  $(x_0)_b$  is the "projection" of a point  $w$  of  $\mathfrak{B}^{(0)}$  contained in  $(w_0)_a$ ;

(b) the set  $\mathfrak{B}^{(0)}$  is connected. (See § 7.)

A *boundary point of a sheet*  $\mathfrak{B}^{(0)}$  is a point not belonging to  $\mathfrak{B}^{(0)}$  but every neighborhood of which contains points of  $\mathfrak{B}^{(0)}$ .

"Sheet" as thus defined corresponds to Bliss's "connected sheet consisting wholly of interior points." This is the only kind we shall have occasion to consider.

Under the hypotheses of Theorem 4, we say that a point  $w = (x, y)$  of  $\mathfrak{B}$  is an *ordinary point* for the function  $G$  if  $w$  is in  $\mathfrak{B}_0$  and the differential  $d_y G(x, y; dy)$  has a reciprocal. In the contrary case we call  $w$  an *exceptional point* for the function  $G$ .

We say that a sheet  $\mathfrak{B}^{(0)}$  is a *sheet of solutions* of the equation (17.3) in case every  $w = (x, y)$  of  $\mathfrak{B}^{(0)}$  satisfies (17.3).

**THEOREM 5.** *If a point  $w_0 = (x_0, y_0)$  is an ordinary point for the function  $G$  and is a solution of the equation (17.3), then there is a unique sheet  $\mathfrak{B}^{(0)}$  of solutions of that equation with the following properties:*

(C<sub>1</sub>)  $\mathfrak{B}^{(0)}$  contains  $w_0$ ;

(C<sub>2</sub>) every point of  $\mathfrak{B}^{(0)}$  is an ordinary point for the function  $G$ ;

(C<sub>3</sub>) the only boundary points of the sheet  $\mathfrak{B}^{(0)}$  are exceptional points for the function  $G$ .

By Theorem 4, there exists at least one sheet of solutions  $\mathfrak{B}^{(1)}$  having properties  $C_1$  and  $C_2$ . Now let  $\mathfrak{B}^{(0)}$  be the least common superclass of all such sheets  $\mathfrak{B}^{(1)}$ . Evidently  $\mathfrak{B}^{(0)}$  is a connected set of solutions satisfying  $C_1$  and  $C_2$ . That  $\mathfrak{B}^{(0)}$  is a sheet follows from  $C_2$  and Theorem 4. To show that  $\mathfrak{B}^{(0)}$  satisfies  $C_3$ , let  $(x_1, y_1) = w_1$  be a boundary point of  $\mathfrak{B}^{(0)}$  and an ordinary point for  $G$ . Since  $G$  would then be continuous at  $(x_1, y_1)$ ,  $G(x_1, y_1) = y_*$ . Then by Theorem 4 we could extend the sheet  $\mathfrak{B}^{(0)}$  to include  $w_1$ , in such a manner that the new sheet satisfies  $C_1$  and  $C_2$ , which contradicts the definition of  $\mathfrak{B}^{(0)}$ . Now suppose that there is a second

sheet  $\mathfrak{B}^{(2)}$  of solutions, having properties  $C_1, C_2, C_3$ . Then  $\mathfrak{B}^{(2)}$  is included in  $\mathfrak{B}^{(0)}$ , and there exists an element  $w_1$  in  $\mathfrak{B}^{(0)}$  but not in  $\mathfrak{B}^{(2)}$ . Since  $\mathfrak{B}^{(0)}$  is connected, there exists a continuous function  $W$  on  $\Re$  to  $\mathfrak{B}^{(0)}$  such that  $W(r_0) = w_0, W(r_1) = w_1, r_0 < r_1$ . By the property  $C_1$  of  $\mathfrak{B}^{(2)}$ ,  $W(r_0)$  is in  $\mathfrak{B}^{(2)}$ . Let  $r_2$  be the least upper bound of the numbers  $r$  in the interval  $r_0 \leq r \leq r_1$  such that  $W(r)$  is in  $\mathfrak{B}^{(2)}$ . Evidently  $W(r_2)$  is a boundary point of  $\mathfrak{B}^{(2)}$ . But since  $W(r_2)$  is in  $\mathfrak{B}^{(0)}$ , it is an ordinary point for  $G$ , which contradicts the property  $C_3$  of  $\mathfrak{B}^{(2)}$ .

Every sheet of solutions determines a single-valued function  $Y(x)$  in a neighborhood of each one of its points. By Theorem 4 each of these functions is of class  $\mathfrak{C}^{(n)}$ .

UNIVERSITY OF MICHIGAN,  
ANN ARBOR, MICH.;  
HARVARD UNIVERSITY,  
CAMBRIDGE, MASS.

# APPLICATION OF THE THEORY OF RELATIVE CYCLIC FIELDS TO BOTH CASES OF FERMAT'S LAST THEOREM\*

(SECOND PAPER)

BY

H. S. VANDIVER

In my first paper under the present title I gave criteria in connection with both cases of the Last Theorem. Here by extensions of the methods previously employed, I shall obtain more general criteria. If

$$(1) \quad x^p + y^p + z^p = 0$$

is satisfied in integers  $x, y$  and  $z$  prime to each other,  $z \not\equiv 0 \pmod{p}$ ,  $p$  an odd prime, then in another paper† I gave the relation

$$(2) \quad \prod_{v=1}^{k-1} \prod_{r=1}^{[vp/k]} (x + \alpha^{[1:r]}y) = \alpha^{-k \cdot vq(k)/(x+y)} \omega^p,$$

where  $k$  is an integer,  $1 < k < p$ ;

$$q(k) = \frac{k^{p-1} - 1}{p},$$

$[s]$  is the greatest integer in  $s$ ;  $\omega$  is an integer in the field  $\Omega(\alpha)$ ;  $\alpha = e^{2\pi i/p}$ ;  $[1:r]$  is the integer  $i$  in the relation  $ri \equiv 1 \pmod{p}$ . Also, throughout the paper, if a fraction  $a/b$  appears as the exponent of  $\alpha$ , it stands for an integer  $u$  which satisfies  $a \equiv bu \pmod{p}$ .

1. Let  $n$  be an odd prime  $\not\equiv 0$  or  $1 \pmod{p}$  and suppose that  $xy \not\equiv 0 \pmod{n}$ ; then

$$(3) \quad x^{n-1} - y^{n-1} \equiv 0 \pmod{n}.$$

If  $\beta$  is a primitive  $(n-1)$ th root of unity then in the field  $\Omega(\alpha\beta)$  we have, since  $n-1$  is prime to  $p$ ,

$$(n) = p_1 p_2 \cdots p_s,$$

where

$$\phi((n-1)p) = ef, \quad n^f \equiv 1 \pmod{(n-1)p}$$

\* Presented to the Society, January 1, 1926; received by the editors February 6, 1926.

† Annals of Mathematics, (2), vol. 21 (1919), p. 78.

the  $\mathfrak{p}$ 's being prime ideals in the field  $\Omega(\alpha\beta)$ , of degree  $f$ . The relation (3) gives

$$\sum_{s=0}^{n-2} (x + \beta^s y) \equiv 0 \pmod{(n)} \\ \equiv 0 \pmod{\mathfrak{p}}$$

hence there is an integer in the set  $0, 1, \dots, n-2$ , such that

$$(4) \quad x + \beta^s y \equiv 0 \pmod{\mathfrak{p}}.$$

It is known that if  $(\theta)$  is an ideal in  $\Omega(\alpha\beta)$  prime to  $(\mathfrak{p})$  and  $\mathfrak{p}, \theta$  an integer in  $\Omega(\alpha\beta)$ , then there is an integer  $s$  such that, if  $w = N(\mathfrak{p}) - 1$ ,

$$\theta^{w/\mathfrak{p}} \equiv \alpha^s \pmod{\mathfrak{p}},$$

where  $N(\mathfrak{p}) = n^f$ , the norm of  $\mathfrak{p}$ . Also  $\theta$  is congruent to the  $\mathfrak{p}$ th power of an integer  $\Omega(\alpha\beta)$  if and only if

$$\left\{ \frac{\theta}{\mathfrak{p}} \right\} = 1, \text{ where } \left\{ \frac{\theta}{\mathfrak{p}} \right\} = \alpha^s, \text{ in general.}$$

Since  $(n)$  is prime to  $(\mathfrak{p})$  then  $\mathfrak{p}$  is prime to  $(\mathfrak{p})$  and  $\mathfrak{p}$  is also prime to  $(x + \alpha^c y)$ ,  $c \not\equiv 0 \pmod{\mathfrak{p}}$ , since the norm of  $x + \alpha^c y$  has all its factors of the form  $1 + w\mathfrak{p}$ . Consequently we may set  $\alpha^m$  for  $\alpha$  in (2),  $m$  any integer  $\not\equiv 0 \pmod{\mathfrak{p}}$ , and take  $\mathfrak{p}$ th power characters of each member of (2) with respect to  $\mathfrak{p}$ , which gives

$$(5) \quad \prod_{r=1}^{k-1} \prod_{s=1}^{[rp/k]} \left\{ \frac{x + \alpha^{m[1:r]} y}{\mathfrak{p}} \right\} = \left\{ \frac{\alpha}{\mathfrak{p}} \right\}^{-mkyq(k)/(x+y)}.$$

We may write

$$\left\{ \frac{x + \alpha^c y}{\mathfrak{p}} \right\} = \left\{ \frac{x + \beta^a y + y(\alpha^c - \beta^a)}{\mathfrak{p}} \right\}$$

and by (4) the right hand member reduces to

$$\left\{ \frac{y(\alpha^c - \beta^a)}{\mathfrak{p}} \right\} = \left\{ \frac{y}{\mathfrak{p}} \right\} \left\{ \frac{\alpha^c - \beta^a}{\mathfrak{p}} \right\}.$$

Now

$$y^{w/\mathfrak{p}} = y^{(n-1)d} \equiv 1 \pmod{(n)},$$

since  $N(\mathfrak{p}) - 1$  is divisible by  $n-1$  but  $n-1$  is prime to  $\mathfrak{p}$ . Hence

$$\left\{ \frac{y}{\mathfrak{p}} \right\} = 1,$$

and therefore

$$\left\{ \frac{x + \alpha^c y}{p} \right\} = \left\{ \frac{\alpha^c - \beta^a}{p} \right\}.$$

Applying this to (5) and using the notation

$$\left\{ \frac{\theta}{p} \right\} = \alpha^{I(\theta)},$$

we have, if we set

$$\sum \text{ for } \sum_{r=1}^{k-1} \sum_{s=1}^{[p/k]},$$

$$(6) \quad \sum I(\alpha^{m[1:r]} - \beta^a) \equiv - \frac{mkyq(k)}{x+y} I(\alpha) \pmod{p}.$$

Set

$$D_s = \sum_{d=1}^{p-1} d^s I(\alpha^d - \beta^a).$$

To determine  $I(\alpha^c - \beta^a)$  in terms of the  $D$ 's, let  $d_1$  be any of the integers 1, 2, ...,  $p-1$  and consider the sum

$$\sum_{s=0}^{p-2} \sum_{d=1}^{p-1} d_1^{p-1-s} d^s I(\alpha^d - \beta^a),$$

which may be put in the form

$$(p-1) I(\alpha^{d_1} - \beta^a) + \sum_{d_1 \neq d} d_1 \frac{d_1^{p-1} - d^{p-1}}{d_1 - d} I(\alpha^d - \beta^a) \pmod{p},$$

whence

$$-I(\alpha^d - \beta^a) \equiv D_0 + d^{p-2} D_1 + d^{p-3} D_2 + \cdots + d D_{p-2},$$

modulo  $p$ . Applying this to (6) we may write, if  $\mu = (p-1)/2$ ,

$$(7) \quad \mu(k-1) D_0 m^{p-1} + \sum (m[1:r])^{p-2} D_1$$

$$+ \sum (m[1:r])^{p-2} D_2 + \cdots + m \left( \sum [1:r] D_{p-2} - \frac{kyq(k)}{x+y} I(\alpha) \right) \equiv 0,$$

modulo  $p$ . Let  $m$  range over the integers 1, 2, ...,  $p-1$ . We obtain, from (7),  $(p-1)$  congruences and since the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^{p-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ p-1 & (p-1)^2 & (p-1)^3 & \cdots & (p-1)^{p-1} \end{vmatrix} = \prod_{i,j=1}^{p-1} (i-j)(p-1)!, \quad i > j,$$

is not divisible by  $p$ , we have, modulo  $p$ ,

$$(8) \quad D_0 \equiv \sum [1:r]^{p-s} D_{s-1} \equiv 0 \pmod{p} \quad (s = 2, 3, \dots, p-2);$$

$$\sum [1:r] D_{p-2} - \frac{kyq(k)}{x+y} I(\alpha) \equiv 0 \pmod{p}.$$

But we also have\*

$$\frac{(1-k^i)b_i}{k^{i-1}i} \equiv \sum [1:r]^{p-i} \pmod{p},$$

where  $b_1 = -\frac{1}{2}$ ,  $b_{2a} = (-1)^{a+1} B_a$ ,  $b_{2a+1} = 0$  ( $a > 0$ ), the  $B$ 's being the numbers of Bernoulli,  $B_1 = 1/6$ ,  $B_2 = 1/30$ , etc. Let  $k$  be a primitive root of  $p$ ; then  $k^l - 1 \not\equiv 0 \pmod{p}$ ,  $l < p-1$ , and also†

$$-kq(k) \equiv \sum [1:r] \pmod{p};$$

and since we may take another value of  $k$  to be  $p-1$ , we have  $q(p-1) \not\equiv 0 \pmod{p}$ , so that (8) becomes, modulo  $p$ , after division by  $k^l - 1$  and  $q(k)$ ,

$$(8a) \quad D_0 \equiv b_{s+1} D_s \equiv 0 \quad (s = 1, 2, \dots, p-3),$$

$$D_{p-2} \equiv -\frac{y}{x+y} I(\alpha).$$

2. Now assume that in (1),  $y$  is divisible by  $p$ ; then it is known that

$$\left( \frac{z + \alpha^l x}{1 - \alpha^l} \right) = q_l^p$$

where  $q_l$  is an ideal in  $\Omega(\alpha)$ ,  $l \not\equiv 0 \pmod{p}$ , and we also have‡

$$\prod q_{[1:A]} \sim 1,$$

\* Annals of Mathematics, (2), vol. 18, p. 114, relation 11.

† Annals of Mathematics, (2), vol. 18, p. 114, relation 12.

‡ Annals of Mathematics, (2), vol. 21 (1919), p. 74.

where  $h$  ranges over the positive integers  $h < p$  which satisfy  $h + |rh| > p$ , or, what is the same thing, the integers  $h$  such that, for  $q = 1, 2, \dots, r$ ,

$$\frac{q}{r+1} < h < \frac{q}{r}, \quad 0 < r < p-1.$$

Here  $|rh|$  is the least positive residue of  $rh$ , modulo  $p$  and  $[1:h]$  stands for the integer  $i$  in  $hi \equiv 1 \pmod{p}$ . Then following the same method employed in the article just cited in deriving (2) of the present paper we find with  $\omega$  an integer in  $\Omega(\alpha)$

$$(9) \quad \prod_h \left( \frac{z + \alpha^{[1:h]}x}{1 - \alpha^{[1:h]}} \right)^{p-1} = \alpha^g \omega^{p(p-1)}$$

where  $g$  is some integer. Let the ideal  $(\rho) = (1 - \alpha)$  and reduce each side of (9) modulo  $(\rho^2)$ . On the left we have

$$\frac{z + \alpha^{[1:h]}x}{1 - \alpha^{[1:h]}} = \frac{z + x}{1 - \alpha^{[1:h]}} - x \equiv -x \pmod{(\rho^2)},$$

since  $z+x$  is divisible by  $p$  and  $(p) = (\rho)^{p-1}$ . Also

$$\alpha^g = (1 - \rho)^g \equiv 1 - g\rho \pmod{(\rho^2)}.$$

Then (9) gives, since  $\omega^{p(p-1)} \equiv 1 \pmod{(\rho^2)}$ ,

$$(-x)^{(p-1)\mu} \equiv 1 - g\rho \pmod{(\rho^2)};$$

or

$$g \equiv 0 \pmod{(\rho)},$$

and since  $g$  is rational,

$$g \equiv 0 \pmod{p},$$

so that (9) may be written in the form

$$(10) \quad \prod_h (z + \alpha^{[1:h]}x)^{p-1} = \prod_h (1 - \alpha^{[1:h]})^{p-1} \omega_1^p, \\ \omega_1 = \omega^{p-1}.$$

Now if  $s$  is one of the  $h$ 's then  $p-s$  is not. Also

$$(1 - \alpha^l) = \alpha^l(\alpha^{-l} - 1)$$

so that

$$\prod_h (1 - \alpha^{[1:h]}) = \alpha^{\sum [1:h]} \prod_h (\alpha^{-[1:h]} - 1).$$

But

$$\prod_h (1 - \alpha^{[1:h]})(\alpha^{-[1:h]} - 1) = (-1)^\mu p,$$

and therefore

$$\prod_h (1 - \alpha^{[1:h]})^{p-1} = (-1)^{\mu^2} p^\mu \alpha^{\mu \Sigma [1:h]}.$$

Then (10) gives

$$(11) \quad \prod_h (z + \alpha^{[1:h]}x)^{p-1} = (-1)^{\mu^2} p^\mu \alpha^{\mu \Sigma [1:h]} \omega_1^p.$$

Now employ the same process on (11) as was used to derive (8a) from (2). Use the same ideal  $\mathfrak{p}$  and take  $p$ th power characters in (11), noting that  $\mathfrak{p}$  is prime to  $(z + \alpha^c x)$ ,  $c \not\equiv 0 \pmod{p}$ . We have

$$\left\{ \frac{p}{\mathfrak{p}} \right\} = 1,$$

since

$$\begin{aligned} p^{w/p} &\equiv p^{(n-1)d} \equiv 1 & (\text{mod } n) \\ &\equiv 1 & (\text{mod } p) \end{aligned}$$

where  $d$  is an integer because  $n-1 \not\equiv 0 \pmod{p}$ . Also

$$\left\{ \frac{-1}{\mathfrak{p}} \right\} = 1$$

since  $N(\mathfrak{p})-1$  is even for  $n$  odd. Hence (11) gives

$$(12) \quad \prod_h \left\{ \frac{z + \alpha^{[1:h]}x}{\mathfrak{p}} \right\}^2 = \left\{ \frac{\alpha}{\mathfrak{p}} \right\}^{\Sigma [1:h]}.$$

Now as in (4) if  $zy \not\equiv 0 \pmod{n}$  there is an integer  $b$  in the set  $0, 1, \dots, n-2$ , such that  $z + \beta^b y \equiv 0 \pmod{p}$ , and (12) gives

$$\prod_h \left\{ \frac{\alpha^{[1:h]} - \beta^b}{\mathfrak{p}} \right\}^2 = \left\{ \frac{\alpha}{\mathfrak{p}} \right\}^{\Sigma [1:h]},$$

or putting  $\alpha^m$  for  $\alpha$  in (11)

$$2 \sum I(\alpha^{m[1:h]} - \beta^b) \equiv m \sum [1:h] I(\alpha) \pmod{p}.$$

In the same way that (7) was obtained we find if

$$\begin{aligned} D'_s &= \sum_{d=1}^{p-1} d^s I(\alpha^d - \beta^b), \\ (13) \quad &\mu D'_0 m^{p-1} + \sum_h (m[1:h])^{p-2} D'_1 + \sum_h (m[1:h])^{p-3} D'_2 + \dots \\ &+ m \left( \sum_h [1:h] D'_{p-2} + \frac{\sum [1:h] I(\alpha)}{2} \right) \equiv 0 \pmod{p}, \end{aligned}$$

and in the same way that (8) was derived we have

$$(13a) \quad \begin{aligned} D'_0 &\equiv \sum [1:h]^{p-s} D'_{s-1} \equiv 0 \pmod{p} \\ (s &= 2, 3, \dots, p-2), \\ \sum [1:h] D'_{p-2} + \frac{\sum [1:h] I(\alpha)}{2} &\equiv 0 \pmod{p}. \end{aligned}$$

But also\*

$$(13b) \quad \sum [1:h]^{p-s} \equiv \frac{b_s(r^{p-s} - (r+1)^{p-s} + 1)}{s} \pmod{p},$$

$$(13c) \quad \sum [1:h] \equiv -rq(r) + (r+1)q(r+1) \pmod{p}.$$

Hence, selecting  $r$  so that  $r^{p-s} - (r+1)^{p-s} + 1 \not\equiv 0 \pmod{p}$ , and proceeding in a similar way with (13c) we obtain

$$(14) \quad \begin{aligned} D'_0 &\equiv 0, \quad b_{s+1} D'_s \equiv 0 \pmod{p} \\ (s &= 1, 2, \dots, p-3); \\ 2D'_{p-2} &\equiv -I(\alpha) \pmod{p}. \end{aligned}$$

3. Consider now the first case of the Last Theorem. The relations (8a) were derived under the assumption that  $xy$  was prime to  $n$ . By assumption  $x$ ,  $y$ , and  $z$  are prime to each other. If one of these is divisible by  $n$  then  $q(n) \equiv 0 \pmod{p}$  by Furtwängler's theorem. If  $a=0$ , or  $(n-1)/2$ , then the congruences  $D_0 \equiv b_{s+1} D_s \equiv 0 \pmod{p}$  all vanish identically, that is, if  $x \pm y \equiv 0 \pmod{n}$ . Of the numbers  $x^2 - y^2$ ,  $x^2 - z^2$ , and  $y^2 - z^2$  select one not divisible by  $p$ , which is always possible. Let  $x^2 - y^2$  be such a number, when, if  $n$  divides  $x^2 - y^2$ , we have  $q(n) \equiv 0 \pmod{p}$  by Furtwängler's theorem. Hence the

THEOREM I. *If*

$$x^p + y^p + z^p = 0$$

*is satisfied in integers none zero and each prime to the odd prime  $p$ , then*

$$q(n)D_0 \equiv 0, \quad q(n)B_{(s+1)/2} D_s \equiv 0 \pmod{p} \quad (s = 1, 3, \dots, p-4);$$

where

$$D_s = \sum_{d=1}^{p-1} d^s I(\alpha^d - \beta^a), \quad \left\{ \frac{\theta}{p} \right\} = \alpha^{I(\theta)},$$

\* Vandiver, *Annals of Mathematics*, (2), vol. 18 (1917), p. 114, relation (13) and the one immediately preceding.

$\mathfrak{p}$  is a prime ideal divisor of  $(n)$  in the field  $\Omega(\alpha\beta)$ ,  $n$  being a rational odd prime,  $\not\equiv 0$  or  $1 \pmod{p}$ ,

$$\alpha = e^{2i\pi/p}; \quad \beta = e^{2i\pi/(n-1)};$$

$\theta$  is any integer in the field  $\Omega(\alpha\beta)$ , such that  $(\theta)$  is prime to  $\mathfrak{p}$ ,  $a$  is some integer in the set  $1, \dots, n-2$ , other than  $(n-1)/2$ ; the  $B$ 's being the numbers of Bernoulli,  $B_1=1/6$ ,  $B_2=1/30$ , etc.

Note that the above criteria are independent of  $x$ ,  $y$  and  $z$ .

Now consider again the relation (8a). If  $z \equiv 0 \pmod{p}$  then  $q(n) \equiv 0 \pmod{p}$ , since  $z \not\equiv 0 \pmod{p}$  by Furtwängler's theorem,\* and we have

**THEOREM II.** *If  $x^p + y^p + z^p = 0$  is satisfied in integers none zero and each prime to the odd prime  $p$ , then*

$$q(n) \sum_{a=1}^{n-2} ((1-v)D_{p-2} + I(\alpha)) \equiv 0 \pmod{p},$$

where  $v$  has any one of the six values  $^1$ ,  $1/t$ ,  $1-l$ ,  $1/(1-l)$ ,  $(l-1)/l$ ,  $t/(l-1)$ ;  $-x/y = ^1$ , the other symbols being defined as in Theorem I.

4. We now will treat the second case of the Last Theorem. Assume in (1) that  $y$  is divisible by  $p$  and that  $xyz \not\equiv 0 \pmod{n}$ ; then (8a) holds with  $D_{p-2} \equiv 0 \pmod{p}$ . If  $xy$  is prime to  $n$  and  $z \equiv 0 \pmod{n}$  then (8a) also holds. Suppose, however, that  $y$  is divisible by  $n$ ; then (14) holds. If  $x \equiv 0 \pmod{n}$  then a set of relations similar to (8a) hold. The relations (8a) and (14) vanish identically, however, if  $a=0$  or  $(n-1)/2$ ; that is, if  $x^2 - y^2$ ,  $z^2 - y^2$  or  $x^2 - z^2 \equiv 0 \pmod{n}$ . Now suppose that  $x+z \not\equiv 0 \pmod{n}$ . If  $x-z \equiv 0 \pmod{n}$  we may employ (8a) instead of (14), since if  $x \pm y \equiv 0 \pmod{n}$  we have  $q(n) \equiv 0 \pmod{p}$ . Whence we have

**THEOREM III.** *If  $p$  is an odd prime and  $x^p + y^p + z^p = 0$  with  $y \equiv 0 \pmod{p}$  and  $xz \not\equiv 0 \pmod{p}$ , then either  $x+z \equiv 0 \pmod{n}$  or*

$$q(n)D_0 \equiv 0, \quad q(n)B_{(s+1)/2}D_s \equiv 0 \pmod{p} \quad (s = 1, 3, \dots, p-4),$$

and in addition one of the two relations

$$q(n)D_{p-2} \equiv 0, \quad q(n)(D_{p-2} + I(\alpha)/2) \equiv 0 \pmod{p},$$

is satisfied, the other symbols being defined as in Theorem I.

\* Wiener Sitzungsberichte, IIa, vol. 121 (1912), pp. 589-92.

In Theorems I and III,  $D_s$  may be shown to be divisible by  $p$  for particular values of  $s$  and  $n$ . For example, if  $n$  is a primitive root of  $p$ , it is easy to show that  $D_s \equiv 0 \pmod{p}$ ,  $s=1, 3, \dots, p-4$ .

From Theorem II of this article it is possible to deduce Theorem I of the first paper under the present title,\* but the proof is obviously much more complicated than that given in the first paper.

5. In all the theorems given here and in the first paper it was assumed that  $n \not\equiv 1 \pmod{p}$ . However it is also possible to give analogous results involving integers  $n$  which are of the form  $1+wp$ . In this case the field  $\Omega(\beta)$  includes  $\Omega(\alpha)$ , and if we go through the same type of argument that was employed to obtain (8a) and (14) we note that  $\{y/p\}$  is not necessarily unity. But we have

$$x+y=v^p,$$

where  $v$  is an integer, whence  $y(1-\beta^\alpha) \equiv v^p \pmod{p}$  and therefore

$$\left\{ \frac{y}{p} \right\} = \left\{ \frac{1-\beta^\alpha}{p} \right\}^{-1}.$$

Also if  $\beta^\alpha$  is a power of  $\alpha$  then  $q(n) \equiv 0 \pmod{p}$ , provided  $y \not\equiv 0 \pmod{p}$ . We then put

$$D_s = \sum_{d=1}^{p-1} d^s I \left( \frac{\alpha^d - \beta^\alpha}{1 - \beta} \right),$$

$\alpha^d \neq \beta^\alpha$ , and proceed as in the proofs in the present paper.

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\*These Transactions, vol. 28 (1926), pp. 554-560.

## RIEMANN INTEGRATION AND TAYLOR'S THEOREM IN GENERAL ANALYSIS\*

BY

LAWRENCE M. GRAVES†

The importance and usefulness of Taylor's theorem need not be dwelt upon here. We are interested in it for functions whose arguments and functional values belong to abstract spaces of the Fréchet type. Consequently no Rolle's theorem can be even stated, and the proofs will be somewhat different from the usual proofs in the theory of numerically-valued functions. It is also to be expected that a slight strengthening of hypotheses will be required. However, it is not necessary to assume the existence of a uniformly continuous  $n$ th differential, as was done in the first announcement of these results.‡ It is sufficient that the function should have an  $n$ th variation (in the sense of Gateaux), with certain limitations on its discontinuities.

The functions we discuss will be one-valued functions whose arguments  $x$  and functional values  $y$  belong to linear metric spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$  respectively, briefly, functions  $F$  on  $\mathfrak{X}$  to  $\mathfrak{Y}$ . Linear metric spaces correspond to the spaces called by Fréchet, "espaces ( $D$ ) vectoriels." For the notations, postulates and fundamental propositions, we refer the reader to Part 1 of a paper by T. H. Hildebrandt and the author,§ to avoid practically entire repetition of that section. No other parts of that paper are needed here.

We shall consistently use the letters  $r, s, a, b, c$  to refer to real numbers, and the German  $\Re$  to refer to the real axis. The notation  $\Re_0$  will then refer to a set of open intervals of that axis. By the notation  $(ab)$  we shall mean the bounded closed interval of  $\Re$  with end points  $a$  and  $b$ . Whenever the space  $\mathfrak{Y}$  is required to be complete, this will be specifically mentioned.

The form of remainder obtained in our Taylor's formula is a generalization of that given by Jordan|| and analogous at least to those obtained in

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† National Research Fellow in Mathematics.

‡ Comptes Rendus, vol. 180 (1925), p. 1719.

§ *Implicit functions and their differentials in general analysis*, in the present number of these Transactions.

|| *Cours d'Analyse*, 3d edition, vol. I, p. 251.

other special cases by Hart,\* and involves a Riemann integral of a function  $G$  on  $\Re$  to a linear metric space  $\mathfrak{Y}$ . Hence it is necessary to develop a theory of Riemann integration for such functions. This is on the whole parallel to, or rather a generalization of, the theory for numerically valued functions.† A theory of line integrals taken along curves in an abstract space  $\Re$  could be developed in a similar way, as well as a theory of Stieltjes integrals. We shall content ourselves here with discussing Riemann integrals.

1. **Derivatives and variations.** For functions  $F$  on  $\Re$  to  $\mathfrak{Y}$  there are several useful definitions of differentiability. We are concerned to base the present theory on the least restrictive one, in order to gain for it the widest range of applicability. We have selected a definition of variations generalizing the one given by Gateaux.‡ Gateaux's definition is less restrictive than the definition of variation used by Lévy in his *Analyse Fonctionnelle*§ and much less restrictive than the definition of differential used by Fréchet.||

**Derivatives.** We say that a function  $F$  on a region  $\Re_0$  of the real axis, to  $\mathfrak{Y}$ , has a derivative, or more specifically, a first derivative,

$$F'(r_0) = \left. \frac{dF}{dr} \right|^{r=r_0}$$

at a point  $r_0$  of  $\Re_0$  in case

$$\lim_{r \rightarrow r_0} \left\| \frac{F(r) - F(r_0)}{r - r_0} - F'(r_0) \right\| = 0.$$

In case  $F$  is defined on a closed interval  $(ab)$ , we define the derivatives at the end points by one-sided limits, as is commonly done in the classical theory of real functions. We define  $n$ th derivatives inductively, as in the classical theory.

\* These Transactions, vol. 18 (1917), p. 138, vol. 23 (1922), p. 39; Annals of Mathematics, (2), vol. 24 (1922), pp. 28; 32.

† N. Wiener has discussed functions  $F$  on  $\mathfrak{C}_0$  to  $\mathfrak{Y}$ , where  $\mathfrak{C}_0$  is a domain of the realm  $\mathfrak{C}$  of complex numbers and  $\mathfrak{Y}$  is a linear metric space having  $\mathfrak{C}$  as its associated number system, and has briefly treated differentiation, integration, and Taylor's series for such functions. See his paper in *Fundamenta Mathematicae*, vol. 4 (1923), p. 136.

‡ Bulletin de la Société Mathématique de France, vol. 47 (1919), p. 83.

§ See pp. 50-52.

|| Cf. these Transactions, vol. 15 (1914), p. 139; vol. 16 (1915), p. 216; Annales Scientifiques de l'École Normale Supérieure, vol. 42 (1925), p. 293. Also § III of the paper on implicit functions by Hildebrandt and Graves, already cited.

**Variations.** We say that a function  $F$  on a region  $\mathfrak{X}_0$  of  $\mathfrak{X}$  to  $\mathfrak{Y}$  has an  $n$ th variation at a point  $x_0$  of  $\mathfrak{X}_0$  in case, for every element  $\delta x$  of  $\mathfrak{X}$ , the function of  $r$ ,  $F(x_0 + r\delta x)$ , has an  $n$ th derivative at  $r=0$ . We denote this  $n$ th derivative at  $r=0$  by  $\delta^n F(x_0, \delta x)$ . Regarded as a function on  $\mathfrak{X}$  to  $\mathfrak{Y}$ ,  $\delta^n F$  is the  $n$ th variation of  $F$  at  $x_0$ . We say that  $F$  has an  $n$ th variation on  $\mathfrak{X}_0$  in case it has an  $n$ th variation at each point of  $\mathfrak{X}_0$ .

**LEMMA 1.1.** *If a function  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  has a first derivative at a point  $r_0$  of  $\mathfrak{X}_0$ , then  $F$  is continuous at  $r_0$ .*

It is not true in general, however, that a function  $F$  on a region  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  which has a first variation is therefore continuous, as numerous examples from the calculus of variations show.

Some properties of the  $n$ th variation (which is readily shown to be unique) are stated in

**LEMMA 1.2.** *Suppose the functions  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$ ,  $G$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$ , and  $H$  on  $\mathfrak{X}_0$  to  $\mathfrak{R}$ , all have  $n$ th variations  $\delta^n F(x_0, \delta x)$ ,  $\dots$ , at a point  $x_0$  of  $\mathfrak{X}_0$ . Then*

(a)  *$F$  has variations of all lower orders at  $x_0$ , and for every positive integer  $k < n$  and every point  $\delta x$  in  $\mathfrak{X}$  we have*

$$\delta^n F(x_0, \delta x) = \frac{d^k}{dr^k} (\delta^{n-k} F(x_0 + r\delta x, \delta x)) \Big|_{r=0};$$

(b) *the sum function  $F+G$  and the product function  $FH$  have  $n$ th variations at  $x_0$ , and  $\delta^n(FH)$  is given by a generalization of Leibnitz's formula;*

(c) *the variations  $\delta^n F$ , etc., are homogeneous of the  $n$ th degree in  $\delta x$ , i. e.,*

$$\delta^n F(x_0, s\delta x) = s^n \delta^n F(x_0, \delta x)$$

*for every point  $\delta x$  of  $\mathfrak{X}$  and every real number  $s$ .*

Property (a) follows at once from the definition of derivative and variation, and property (b) is proved in the usual way, considering derivatives first. To prove property (c), we proceed by induction. We have first, if  $s \neq 0$ ,

$$\begin{aligned} \|\delta F(x_0, s\delta x) - s\delta F(x_0, \delta x)\| &\leq \left\| \delta F(x_0, s\delta x) - \frac{F(x_0 + (r/s)s\delta x) - F(x_0)}{(r/s)} \right\| \\ &+ |s| \left\| \frac{F(x_0 + r\delta x) - F(x_0)}{r} - \delta F(x_0, \delta x) \right\|. \end{aligned}$$

Since the terms on the right approach zero with  $r$ , the left hand side equals zero. The case  $s=0$  follows from the fact that  $\delta F(x_0, x_*) = y_*$  for every function  $F$ . The manipulation to complete the induction proceeds in the same way as above.

2. **Riemann integration.** The theory of Riemann integration of bounded functions here presented is on the whole parallel to the classical theory. A noteworthy hiatus is the failure to show that for the existence of the integral it is *necessary* that the Lebesgue measure of the set of discontinuities of the integrand shall be zero. That this condition is not necessary is shown by the following example, where the integrand function is discontinuous at every point of the interval  $0 \leq r \leq 1$ . Let the space  $\mathcal{Y}$  consist of all functions  $y$  on the interval  $0 \leq t \leq 1$  to  $\mathcal{R}$  which are bounded on that interval, and let  $\|y\|$  = the upper bound of  $|y(t)|$ . Consider the system of points  $y_r$  defined by

$$y_r(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq r, \\ 1 & \text{for } r \leq t \leq 1, \end{cases} \quad 0 \leq r \leq 1.$$

Let  $F(r) = y_r$ . Then  $F$  is integrable and yet everywhere discontinuous.

**Definition of integrals.** Consider a bounded function  $F$  on  $(ab)$  to  $\mathcal{Y}$ . (By bounded we mean that  $\|F(r)\|$  is bounded on  $(ab)$ ). Let  $\pi$  be a partition of  $(ab)$  into sub-intervals  $\Delta_i$  of lengths  $\Delta_i$ . Denote the norm of the partition by  $N\pi$ . The lengths  $\Delta_i$  are understood to have the same sign as  $(b-a)$ . Let  $r_i$  be an arbitrary point of the interval  $\Delta_i$ . Then if the limit

$$\lim_{N\pi=0} \sum_{\pi} F(r_i) \Delta_i$$

exists, we say that  $F$  is integrable on  $(ab)$ , and denote the limit by the usual symbol

$$\int_a^b F(r) dr.$$

The limit is taken in the sense that, for every positive  $\epsilon$  there exists a positive  $\delta$  such that, for every partition  $\pi$  with  $N\pi \leq \delta$  and every choice of the points  $r_i$  in the intervals  $\Delta_i$  of  $\pi$ , we have

$$\left\| \sum_{\pi} F(r_i) \Delta_i - \int_a^b F(r) dr \right\| \leq \epsilon.$$

In a complete linear metric space we have the usual necessary and sufficient condition for the existence of the integral, stated in

LEMMA 2.1 *If the space  $\mathfrak{Y}$  is complete, then a necessary and sufficient condition that a bounded function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  be integrable on  $(ab)$ , is that*

$$\lim_{\substack{N\pi_1=0 \\ N\pi_2=0}} \left\| \sum_{\pi_1} F(r_{i1})\Delta_{i1} - \sum_{\pi_2} F(r_{i2})\Delta_{i2} \right\| = 0.$$

We find it convenient in proving Theorem 1, on the sufficiency of certain conditions for the existence of the integral, to derive the lemmas numbered 2.2 and 2.3 relating to oscillation and the sets of discontinuities of functions.

**Oscillation.** For an interval  $(ab)$  and a function  $F$  on  $(ab)$  to  $\mathfrak{Y}$ , we define the oscillation  $O_F(a, b)$  to be the upper bound of  $\|F(r_1) - F(r_2)\|$  for  $r_1$  and  $r_2$  in  $(ab)$ . For an interior point  $c$  of  $(ab)$  we define the point oscillation  $O_F(c)$  by the equation

$$O_F(c) = \lim_{r=0} O_F(c-r, c+r).$$

If  $c$  is an end point, say the left hand end point, of the interval, then we put  $c$  in place of  $(c-r)$  in the limitand. The oscillation functions  $O_F(a, b)$  and  $O_F(r)$  are evidently always well defined for bounded functions  $F$ . A necessary and sufficient condition for the continuity of a function  $F$  at a point  $c$  is that  $O_F(c) = 0$ .

**Content and measure.** Let  $\mathfrak{E}$  be a set of points of the real axis  $\mathfrak{R}$ . Then we shall mean by *content*  $\mathfrak{E}$ , the Jordan measure of  $\mathfrak{E}$ ,\* and by *measure*  $\mathfrak{E}$ , the Lebesgue measure of  $\mathfrak{E}$ ,\* if these measures exist.

For a function  $F$  on  $(ab)$  to  $\mathfrak{Y}$ , we shall denote by  $\mathfrak{E}_{\epsilon F}$  the set of points  $r$  of  $(ab)$  at which  $O_F(r) \geq \epsilon$ , and by  $\mathfrak{D}_F$  the set of points of  $(ab)$  at which  $F$  is discontinuous.

LEMMA 2.2. *For every bounded function  $F$  on  $(ab)$  to  $\mathfrak{Y}$ , the statements "content  $\mathfrak{E}_{\epsilon F} = 0$  for every  $\epsilon > 0$ ," and "measure  $\mathfrak{D}_F = 0$ " are equivalent.*

Consider a sequence of positive numbers  $\{\epsilon_k\}$  with  $\lim \epsilon_k = 0$ . Then  $\mathfrak{D}_F = \sum_1^\infty \mathfrak{E}_{\epsilon_k F}$ . By definition of content, each  $\mathfrak{E}_{\epsilon_k F}$  is enclosable interiorly in a finite number of non-overlapping intervals the sum of whose lengths is less than  $\epsilon/2^k$ , where  $\epsilon$  is arbitrary. Hence  $\mathfrak{D}_F$  is enclosable in a denumerable infinity of intervals the sum of whose lengths is arbitrarily small, so that measure  $\mathfrak{D}_F = 0$ . For the converse, we prove first that every set  $\mathfrak{E}_{\epsilon F}$  is closed. Let  $c$  be a limit point of  $\mathfrak{E}_{\epsilon F}$ . Then every interval  $(c-r, c+r)$  encloses a point of  $\mathfrak{E}_{\epsilon F}$ , so that  $O_F(c-r, c+r) \geq \epsilon$  for every positive  $r$ , and

\*See Jordan, *Cours d'Analyse*, 3d edition, vol. I, p. 28; Lebesgue, *Leçons sur l'Intégration*, p. 102 and p. 28.

hence  $O_F(c) \geq \epsilon$ . Now by definition of "measure  $\mathfrak{D}_F = 0$ ," for every positive number  $\omega$  there exists a denumerable set of intervals enclosing  $\mathfrak{D}_F$  interiorly with the sum of their lengths less than  $\omega$ . Since each set  $\mathfrak{G}_{\epsilon F}$  is a part of the set  $\mathfrak{D}_F$  and since each  $\mathfrak{G}_{\epsilon F}$  is closed, we can apply the Heine-Borel-Lebesgue theorem in generalized form\* to show that each  $\mathfrak{G}_{\epsilon F}$  is enclosed interiorly by a finite number of intervals the sum of whose lengths is less than  $\omega$ . Since  $\omega$  is arbitrary, each  $\mathfrak{G}_{\epsilon F}$  has content zero.

LEMMA 2.3. *Let  $F$  be a function on  $(ab)$  to  $\mathfrak{Y}$  such that  $O_F(r) \leq \epsilon$  on  $(ab)$ . Then for every constant  $\omega > 0$  there exists a constant  $\delta > 0$  such that, for every pair  $\pi_1, \pi_2$  of partitions of  $(ab)$  with norms less than  $\delta$  we have*

$$(2.1) \quad \left\| \sum_{\pi_1} F(r_{i1}) \Delta_{i1} - \sum_{\pi_2} F(r_{i2}) \Delta_{i2} \right\| \leq (\epsilon + \omega) |b - a|.$$

The hypothesis  $O_F(r) \leq \epsilon$  on  $(ab)$  implies that  $F$  is bounded on  $(ab)$ . By definition of the point oscillation  $O_F(r)$ , we know that if we fix  $\omega$  arbitrarily, then for every point  $r$  of  $(ab)$  there is a constant  $2\alpha_r$  such that

$$(2.2) \quad O_F(r - 2\alpha_r, r + 2\alpha_r) \leq \epsilon + \omega.$$

Since  $(ab)$  is a closed interval, by the Heine-Borel-Lebesgue theorem there is a finite subset  $(r_i - \alpha_{r_i}, r_i + \alpha_{r_i})$  of the set of intervals  $(r - \alpha_r, r + \alpha_r)$ , which also covers  $(ab)$ . Let  $4\delta$  be the minimum length of the intervals of this finite subset, i. e.,  $2\delta = \text{minimum } \alpha_{r_i}$ . Then if  $\pi_1$  and  $\pi_2$  have norms less than  $\delta$ , and  $\Delta_1$  and  $\Delta_2$  are intervals of  $\pi_1$  and  $\pi_2$  respectively having a point in common, there is an interval  $(r_i - 2\alpha_{r_i}, r_i + 2\alpha_{r_i})$  to which  $\Delta_1$  and  $\Delta_2$  are both interior. Let  $\pi_3$  be the partition of  $(ab)$  obtained by using all the division points of  $\pi_1$  and  $\pi_2$ . Then by the inequality (2.2) we have

$$\begin{aligned} \left\| \sum_{\pi_1} F(r_{i1}) \Delta_{i1} - \sum_{\pi_2} F(r_{i2}) \Delta_{i2} \right\| &= \left\| \sum_{\pi_3} [F(r_{k1}) - F(r_{k2})] \Delta_{k3} \right\| \\ &\leq (\epsilon + \omega) |b - a|. \end{aligned}$$

THEOREM 1. *Existence theorem. Suppose that the space  $\mathfrak{Y}$  is complete, and that the function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  is bounded on  $(ab)$ , and has measure  $\mathfrak{D}_F = 0$ . Then  $F$  is integrable on  $(ab)$ .*

\*For a simple proof of this theorem, see Lebesgue, loc. cit., p. 105. Only a slight modification of Lebesgue's reasoning is necessary to prove the theorem when the interval  $(ab)$  is replaced by an arbitrary bounded closed set  $E$ . See Hildebrandt, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 423 ff.

By Lemma 2.2, we may replace the hypothesis "measure  $\mathfrak{D}_F=0$ " by "content  $\mathfrak{E}_{\epsilon F}=0$  for every positive  $\epsilon$ ." Select positive numbers  $\epsilon$ ,  $\omega$ , and  $\alpha$  arbitrarily. Since content  $\mathfrak{E}_{\epsilon F}=0$ , there is a finite set  $\mathfrak{S}$  composed of  $n$  intervals, the sum of whose lengths is less than  $\alpha$ , and which enclose the set  $\mathfrak{E}_{\epsilon F}$  in their interiors. Let  $\mathfrak{T}$  be the set of intervals complementary to  $\mathfrak{S}$  on  $(ab)$ . Then  $O_F(r) < \epsilon$  on  $\mathfrak{T}$ . By Lemma 2.3, there is a positive  $\delta_\omega$  corresponding to  $\omega$  such that the inequality (2.1) holds for all partitions  $\pi_1$  and  $\pi_2$  of the set of intervals  $\mathfrak{T}$ , having norm less than  $\delta_\omega$ . Select  $\delta_1 < \delta_\omega$  and  $< \alpha/n$ , and let  $\pi_3$  and  $\pi_4$  be partitions of  $(ab)$  of norm less than  $\delta_1$ . Let  $\pi_5$  and  $\pi_6$  be the partitions formed from  $\pi_3$  and  $\pi_4$  respectively by inserting the end points of the intervals of the set  $\mathfrak{S}$  as division points. The number of intervals of  $\pi_3$  which are not identical with intervals of  $\pi_5$  is not greater than  $2n$ , where  $n$  is the number of intervals in  $\mathfrak{S}$ . We specify that on an interval  $\Delta_{j5}$  identical with a  $\Delta_{i3}$ ,  $r_{j5}=r_{i3}$ , so that

$$\left\| \sum_{\pi_3} F(r_{i3}) \Delta_{i3} - \sum_{\pi_5} F(r_{j5}) \Delta_{j5} \right\| \leq 4Mn\delta_1 \leq 4M\alpha,$$

where  $M$  is the upper bound of  $\|F(r)\|$  on  $(ab)$ . We do similarly for  $\pi_4$  and  $\pi_6$ . Next we have

$$\left\| \sum_{\pi_5} F(r_{j5}) \Delta_{j5} \right\| + \left\| \sum_{\pi_6} F(r_{j6}) \Delta_{j6} \right\| \leq 2M\alpha,$$

where the sums are taken over the intervals of  $\pi_5$  and  $\pi_6$  contained in the set  $\mathfrak{S}$ . Since  $\delta_1 < \delta_\omega$ , we have finally

$$\left\| \sum_{\pi_5} F(r_{j5}) \Delta_{j5} - \sum_{\pi_6} F(r_{j6}) \Delta_{j6} \right\| \leq (\epsilon + \omega) |b - a|.$$

By combination of these inequalities we obtain

$$\left\| \sum_{\pi_3} F(r_{i3}) \Delta_{i3} - \sum_{\pi_4} F(r_{i4}) \Delta_{i4} \right\| \leq 10M\alpha + (\epsilon + \omega) |b - a|.$$

Since  $\alpha$ ,  $\epsilon$  and  $\omega$  are arbitrary, we may apply Lemma 2.1 to obtain the desired conclusion.

The next theorem contains the ordinary formulas for definite integrals, which extend immediately to our general case.

THEOREM 2. Suppose that the functions  $F$ ,  $G$ , and  $H$ , on  $(ab)$  to  $\mathfrak{Y}$ ,  $\mathfrak{V}$ , and  $\Re$  respectively, are integrable on  $(ab)$ . Let  $c$ ,  $d$ , and  $e$  be points of  $(ab)$ . Then the following equalities and inequality are valid:

$$(2.3) \quad \int_a^d F dr + \int_d^e F dr = \int_a^e F dr \text{ if the space } \mathfrak{Y} \text{ is complete ;}$$

$$(2.4) \quad \int_a^b (F + G) dr = \int_a^b F dr + \int_a^b G dr ;$$

$$(2.5) \quad \int_a^b FH dr = H \int_a^b F dr \text{ if } H \text{ is constant on } (ab) ;$$

$$(2.6) \quad \int_a^b FH dr = F \int_a^b H dr \text{ if } F \text{ is constant on } (ab) ;$$

$$(2.7) \quad \left\| \int_a^b F dr \right\| \leq \left| \int_a^b H dr \right| \text{ if } \|F(r)\| \leq H(r) \text{ on } (ab).$$

3. Relations between derivatives and integrals. We say that a function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  has a *primitive*  $H$  in case the function  $H$  is defined on  $(ab)$  to  $\mathfrak{Y}$  and has  $F$  for its derivative on  $(ab)$ . If a function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  is integrable on every sub-interval of  $(ab)$ , then the function

$$G(r) = \int_a^r F(r) dr$$

is called an *indefinite integral* of  $F$ . In Theorem 4 we show essentially that when a function  $F$  has both a primitive and an indefinite integral, these differ at most by a constant element of the space  $\mathfrak{Y}$ . A particular case where both a primitive and an indefinite integral exist is when the space  $\mathfrak{Y}$  is complete, and the function  $F$  is continuous. This follows from Theorems 1 and 3. Lemma 3.1 is concerned with integration by parts.

THEOREM 3. If the space  $\mathfrak{Y}$  is complete, and if the function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  is integrable on  $(ab)$ , then the function  $G$  on  $(ab)$  to  $\mathfrak{Y}$  defined by

$$G(r) = \int_a^r F dr$$

has the properties

(1)  $\|G(r_1) - G(r_2)\| \leq M|r_1 - r_2|$  for every  $r_1, r_2$  in  $(ab)$ , where  $M$  is the upper bound of  $\|F(r)\|$  on  $(ab)$ ;

- (2)  $G$  is continuous on  $(ab)$ ;  
 (3) if  $F$  is continuous at a point  $r_0$  of  $(ab)$ , then  $G$  has a derivative at  $r_0$ , equal to  $F(r_0)$ .

This theorem is proved by application of the formulas of Theorem 2.

THEOREM 4. If the function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  has a derivative  $F'$  on  $(ab)$  which is integrable on  $(ab)$ , then

$$F(b) - F(a) = \int_a^b F' dr. *$$

For definiteness we assume  $a < b$ . Let  $\epsilon$  be an arbitrary positive number. Then since  $F'$  is integrable, there exists a  $\delta > 0$  such that, for every partition  $\pi$  with  $N\pi \leq \delta$  and for every choice of the  $r_i$  in the closed intervals  $\Delta_i$  of  $\pi$ , we have

$$(3.1) \quad \left\| \sum_{\pi} F'(r_i) \Delta_i - \int_a^b F' dr \right\| \leq \epsilon.$$

Since  $F'$  is the derivative of  $F$  on  $(ab)$ , there is for each point  $r$  of  $(ab)$  a positive number  $\alpha_r \leq \delta$  such that, for every point  $r'$  of  $(ab)$  satisfying  $|r' - r| \leq \alpha_r$ , we have

$$(3.2) \quad \|F(r') - F(r) - F'(r)(r' - r)\| \leq \epsilon |r' - r|.$$

The open intervals  $I_r \equiv (r - \alpha_r, r + \alpha_r)$  constitute a set covering the closed interval  $(ab)$ ; hence we may apply the Heine-Borel-Lebesgue theorem to show that a finite set  $I_1 \cdots I_m$  of these, with centers at  $\rho_1 < \rho_2 < \cdots < \rho_m$  respectively, also cover  $(ab)$ . We may evidently assume that no one of the intervals  $I_k$  is wholly contained in another one of them, and that  $\rho_1 = a$ ,  $\rho_m = b$ . For each interval  $I_k$  we have by (3.2) the condition that when  $r$  is in  $I_k$  or at an end point of  $I_k$ ,

$$(3.3) \quad \|F(r) - F(\rho_k) - F'(\rho_k)(r - \rho_k)\| \leq \epsilon |r - \rho_k|.$$

Now in the interval  $\rho_k \leq r \leq \rho_{k+1}$  blot out all end points of intervals  $I_j$  having  $j \neq k$ . The remaining end points together with the points  $\rho_k$ , determine a partition  $\pi$  of  $(ab)$  of norm  $\leq \delta$ . Each interval  $\Delta_i = (r_i, r_{i+1})$  of  $\pi$  has at least one of the points  $\rho_k$  for an end point. If there are two, we choose the left hand one, and in either case denote the  $\rho_k$  thus determined for  $\Delta_i$  by  $\rho_i$ . Then by (3.3) we have for every  $i$ ,

$$(3.4) \quad \|F(r_{i+1}) - F(r_i) - F'(\rho_i) \Delta_i\| \leq \epsilon \Delta_i.$$

\*This theorem in its present generality is due to T. H. Hildebrandt, as is also the second corollary. I have slightly altered his proof.

Thus by (3.1) and (3.4) we obtain

$$\begin{aligned} \left\| F(b) - F(a) - \int_a^b F' dr \right\| &\leq \sum_i \left\| F(r_{i+1}) - F(r_i) - F'(\rho_i) \Delta_i \right\| \\ &+ \left\| \sum_i F'(\rho_i) \Delta_i - \int_a^b F' dr \right\| \leq \epsilon(b-a+1). \end{aligned}$$

Since  $\epsilon$  is arbitrary, the equality is proved.

That two primitives of a function always differ by a constant is a result of

**COROLLARY 1.** *If the function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  has a derivative  $F'(r) = y_*$  on  $(ab)$ , then  $F$  is constant on  $(ab)$ .\**

For the function  $F'$  is evidently integrable on every sub-interval of  $(ab)$ , and hence

$$F(r) - F(a) = \int_a^r F' dr = y_*$$

for every  $r$  in  $(ab)$ .

**COROLLARY 2.** *If the space  $\mathfrak{Y}$  is complete, and if the function  $F$  on  $(ab)$  to  $\mathfrak{Y}$  has a derivative  $F'$  on  $(ab)$  which is continuous on  $(ab)$ , then*

$$\lim_{r_1=r_2} \left\| \frac{F(r_1) - F(r_2)}{r_1 - r_2} - F'(r_2) \right\| = 0$$

*uniformly on  $(ab)$ .*

Since  $F'$  is continuous on  $(ab)$ , it is bounded on  $(ab)$ , and since  $(ab)$  is closed,  $F'$  is continuous uniformly on  $(ab)$ . This can be shown by classical methods. Then by applying Theorems 1, 4, and 2, we obtain

$$\left\| \frac{F(r_1) - F(r_2)}{r_1 - r_2} - F'(r_2) \right\| = \left\| \frac{\int_{r_2}^{r_1} [F'(r) - F'(r_2)] dr}{r_1 - r_2} \right\| \leq \epsilon$$

whenever  $|r_1 - r_2|$  is sufficiently small.

**LEMMA 3.1.** *Integration by parts. Suppose the space  $\mathfrak{Y}$  is complete, and that the functions  $F$  on  $(ab)$  to  $\mathfrak{Y}$  and  $G$  on  $(ab)$  to  $\mathfrak{R}$  have derivatives  $F'$  and  $G'$  on  $(ab)$  which are bounded on  $(ab)$  and whose sets of discontinuities on  $(ab)$  each have measure zero. Then*

$$F(b)G(b) - F(a)G(a) = \int_a^b F(r)G'(r)dr + \int_a^b F'(r)G(r)dr.$$

\* Cf. Fréchet, *Annales de l'École Normale Supérieure*, vol. 42 (1925), pp. 313, 316.

For the proof we apply Lemmas 1.2 and 1.1 and Theorems 1, 4, and 2.

4. **Taylor's theorem.** Taylor's theorem is valid only for convex regions as defined below. However, it should be remarked that in a linear metric space every "neighborhood" constitutes a convex region.

**Convex regions.** A region  $\mathfrak{X}_0$  of a linear metric space  $\mathfrak{X}$  is *convex* in case, for every pair of points  $x_1, x_2$  of  $\mathfrak{X}_0$  and every number  $r$  in the interval  $(0, 1)$ , the point  $x_1 + (x_2 - x_1)r$  is in  $\mathfrak{X}_0$ .

**THEOREM 5. TAYLOR'S THEOREM.** *Suppose that the space  $\mathfrak{Y}$  is complete, and that the region  $\mathfrak{X}_0$  of the space  $\mathfrak{X}$  is convex, and suppose that the function  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  has an  $n$ th variation on  $\mathfrak{X}_0$ . Suppose also that for every  $x_1, x_2$  in  $\mathfrak{X}_0$  the function of  $r$ ,  $\delta^n F(x_1 + (x_2 - x_1)r, x_2 - x_1)$ , is bounded on the interval  $(0, 1)$  and its set of discontinuities is of measure zero. Then for every  $x_1, x_2$  in  $\mathfrak{X}_0$  we have*

$$F(x_2) = F(x_1) + \sum_{i=1}^{n-1} \delta^i F(x_1, x_2 - x_1)/i! + R_n(x_1, x_2)$$

where

$$R_n(x_1, x_2) = \int_0^1 \delta^n F(x_1 + (x_2 - x_1)r, x_2 - x_1) \frac{(1-r)^{n-1}}{(n-1)!} dr.$$

We first note the fact that for  $n > 1$ , the existence of the  $n$ th variation of  $F$  on  $\mathfrak{X}_0$  implies the continuity and hence the boundedness of the function  $\delta^{n-1}F(x_1 + (x_2 - x_1)r, x_2 - x_1)$  on the interval  $(0, 1)$ , by Lemma 1.1. Consider now the case  $n = 1$ . The function  $F(x_1 + (x_2 - x_1)r)$  has a first derivative  $\delta F(x_1 + (x_2 - x_1)r, x_2 - x_1)$  which is integrable on  $(0, 1)$ , by definition of variation and Theorem 1. Then Theorem 4 yields the required result. Now assume the formula true for  $n = m$ , and that  $F$  has an  $(m+1)$ st variation with the specified properties. Then the function  $G(r) = \delta^m F(x_1 + (x_2 - x_1)r, x_2 - x_1)$  has for its derivative  $\delta^{m+1}F(x_1 + (x_2 - x_1)r, x_2 - x_1)$ . The function  $H(r) = -(1-r)^m/m!$  has a continuous derivative  $H'(r)$ . Therefore we may integrate the remainder  $R_m$  by parts (Lemma 3.1) and obtain

$$R_m(x_1, x_2) = \int_0^1 G(r)H'(r)dr = \frac{\delta^m F(x_1, x_2 - x_1)}{m!} + R_{m+1}(x_1, x_2).$$

This completes the induction.

Note that by Lemma 1.2, the Taylor's expansion has the following properties:

$$(a) \quad \delta^i F(x, (\delta x)r) = \delta^i F(x, \delta x)r^i \quad (i = 1, \dots, n-1),$$

and

$$(b) \quad \lim_{s \rightarrow 0} R_n(x, x + (\delta x)s)/s^{n-1} = y_*,$$

holding for every  $x$  in  $\mathfrak{X}_0$ ,  $\delta x$  in  $\mathfrak{X}$ , and  $r$  in  $\mathfrak{R}$ . The uniqueness of an expansion having these properties is shown in

**THEOREM 6.** *Let the function  $F$  be on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$ , and let  $x_0$  be a point of  $\mathfrak{X}_0$ . Then for each positive integer  $m$  there is not more than one expansion*

$$(4.1) \quad F(x) = F(x_0) + \sum_{i=1}^m L_i(x - x_0) + R_{m+1}(x)$$

having the properties

$$(4.2) \quad L_i((\delta x)r) = L_i(\delta x)r^i \quad (i = 1, \dots, m),$$

$$(4.3) \quad \lim_{s \rightarrow 0} \frac{R_{m+1}(x_0 + (\delta x)s)}{s^m} = y_*,$$

holding for every real number  $r$  and every  $\delta x$  in  $\mathfrak{X}$ .

For each  $\delta x$  of  $\mathfrak{X}$ , the point  $x_0 + (\delta x)s$  is in  $\mathfrak{X}_0$  when  $s$  is sufficiently small. Now assume two expansions (4.1) with terms  $L_i$  and  $M_i$  and remainders  $R_{m+1}$  and  $S_{m+1}$  respectively. Assume also that for a certain integer  $k$  satisfying  $1 \leq k \leq m$  we have

$$(4.4) \quad L_i = M_i \quad (1 \leq i < k).$$

Then by substituting  $x_0 + (\delta x)s$  for  $x$  in (4.1) and using property (4.2) we obtain the inequality

$$\begin{aligned} \|L_k(\delta x) - M_k(\delta x)\| &\leq \sum_{i=k+1}^m \left[ \|L_i(\delta x)\| + \|M_i(\delta x)\| \right] |s|^{i-k} \\ &\quad + \frac{\|R_{m+1}(x_0 + (\delta x)s)\|}{|s|^k} + \frac{\|S_{m+1}(x_0 + (\delta x)s)\|}{|s|^k}. \end{aligned}$$

By property (4.3), the right hand side approaches zero with  $s$ . Therefore  $L_k = M_k$  is a consequence of the assumption (4.4). By  $m$  applications of this result we arrive at the desired conclusion.

**5. Applications.** We shall discuss here only a few simple applications of Taylor's theorem. As a first consequence we state an existence theorem for the expansion whose uniqueness was shown in Theorem 6. We have already noted that the existence of such an expansion follows immediately from Taylor's theorem, but by a simple manipulation it is possible to show that existence under less restrictive hypotheses.

**THEOREM 7.** Let the space  $\mathfrak{Y}$  be complete, and let  $x_0$  be a point of the space  $\mathfrak{X}$ . Let  $F$  be a function defined on a neighborhood  $(x_0)_a$  to  $\mathfrak{Y}$ , which has an  $(m-1)$ st variation on  $(x_0)_a$  satisfying the hypotheses of Taylor's theorem, and which has also an  $m$ th variation at  $x_0$ . Then the function  $R_{m+1}$  on  $(x_0)_a$  to  $\mathfrak{Y}$  defined by the equation

$$F(x) = F(x_0) + \sum_{i=1}^m \delta^i F(x_0, x - x_0)/i! + R_{m+1}(x)$$

satisfies the condition

$$\lim_{r \rightarrow 0} \frac{R_{m+1}(x_0 + \delta x r)}{r^m} = y_*$$

for every  $\delta x$  of  $\mathfrak{X}$ .\*

For the case  $m=1$  we omit the hypothesis concerning  $\delta^{m-1}F$ . The conclusion in this case follows immediately from the definition of variation. For  $m>1$ , we apply Taylor's theorem with  $n=m-1$  to obtain the formula, valid for every fixed  $\delta x$  in  $\mathfrak{X}$ , for every real value of  $t$  sufficiently small,

$$R_{m+1}(x_0 + (\delta x)t) = \int_0^1 S(r, t) \frac{r(1-r)^{m-2}}{(m-2)!} t^m dr,$$

where

$$S(r, t) = \frac{\delta^{m-1}F(x_0 + (\delta x)tr, \delta x) - \delta^{m-1}F(x_0, \delta x)}{tr} \\ - \delta^m F(x_0, \delta x) \quad (tr \neq 0), \\ S(r, t) = y_* \quad (tr = 0).$$

By definition of  $m$ th variation, we have

$$\lim_{t \rightarrow 0} S(r, t) = y_*$$

uniformly on  $0 \leq r \leq 1$ . Then application of Theorem 2 yields the required result.

**Symmetry of differentials.**† Symmetry has no meaning for variations as we have defined them. But for differentials of order higher than the first, as defined by Fréchet‡ and by Hildebrandt and Graves in the paper previously cited, Part III, we do have a theorem on symmetry. This is a

\* In discussing these theorems for the case  $\mathfrak{X}=\mathfrak{Y}=\mathfrak{R}$ , W. H. Young states, in the Proceedings of the London Mathematical Society, vol. 7 (1909), p. 158, that the existence of an expansion of  $F$  at  $x_0$  in the form given in Theorem 6 is equivalent to the existence of an  $m$ th derivative of  $F$  at  $x_0$ . That this is erroneous is shown by a simple example such as  $F(x) = x^2 \sin(1/x)$ ,  $m=2$ .

† Cf. Fréchet, these Transactions, vol. 16 (1915), p. 233.

‡ Annales de l'École Normale Supérieure, vol. 42 (1925), p. 321.

generalization of the theorem on inversion of the order of partial differentiation for ordinary functions of several variables. For present purposes it is sufficient to define differentials as follows. We say that a function  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  has a *first differential* at a point  $x_0$  of  $\mathfrak{X}_0$  in case there exists a function  $dF(x_0, dx)$  on  $\mathfrak{X}$  to  $\mathfrak{Y}$  with the following properties:

(1)  $dF(x_0, (d_1x)a_1 + (d_2x)a_2) = dF(x_0, d_1x)a_1 + dF(x_0, d_2x)a_2$  for every pair  $d_1x, d_2x$  in  $\mathfrak{X}$  and every pair of numbers  $a_1, a_2$ ;

(2) for every  $\epsilon > 0$  there exists a  $d > 0$  such that, for every  $\Delta x$  in  $\mathfrak{X}$  such that  $\|\Delta x\| \leq d$  we have

$$\|F(x_0 + \Delta x) - F(x_0) - dF(x_0, \Delta x)\| \leq \|\Delta x\|\epsilon.$$

We say that  $F$  has an  *$n$ th differential* at  $x_0$  in case  $F$  has an  $(n-1)$ st differential  $d^{n-1}F(x, d_1x, \dots, d_{n-1}x)$  which is continuous in  $x$  in a neighborhood of  $x_0$ , and the function  $d^{n-1}F$  has a first differential at  $x_0$  for each  $d_1x, \dots, d_{n-1}x$  in  $\mathfrak{X} \cdots \mathfrak{X}$ . We note that a function which has an  $n$ th differential at a point certainly has an  $n$ th variation at that point, and that the  $n$ th differential if continuous satisfies the requirements of Taylor's theorem on the  $n$ th variation. The above definition requires less than the one given by Fréchet, but is sufficient to validate the following theorem on symmetry.

**THEOREM 8.** *If the space  $\mathfrak{Y}$  is complete, and the function  $F$  on  $\mathfrak{X}_0$  to  $\mathfrak{Y}$  has an  $n$ th differential at  $x_0$  ( $n \geq 2$ ), then  $d_n F(x_0, d_1x, \dots, d_nx)$  is symmetric in each pair of differentials  $d_ix, d_jx$ , of the independent variable  $x$ .*

Consider first the case  $n=2$ . Since  $F$  has a continuous first differential on a neighborhood of  $x_0$ , we can apply Taylor's theorem on such a neighborhood to obtain

$$\begin{aligned} H(a) - d^2F(x_0, d_1x, d_2x) &= \frac{F(x_0 + (d_1x)a + (d_2x)a) - F(x_0 + (d_2x)a)}{a^2} \\ &\quad - \frac{F(x_0 + (d_1x)a) - F(x_0)}{a^2} - d^2F(x_0, d_1x, d_2x) \\ &= \int_0^1 \left[ \frac{dF(x_0 + (d_1x)ar + (d_2x)a, d_1x) - dF(x_0, d_1x)}{a} \right. \\ &\quad \left. - d^2F(x_0, d_1x, (d_1x)r + d_2x) \right] dr \\ &\quad - \int_0^1 \left[ \frac{dF(x_0 + (d_1x)ar, d_1x) - dF(x_0, d_1x)}{a} \right. \\ &\quad \left. - d^2F(x_0, d_1x, (d_1x)r) \right] dr. \end{aligned}$$

Since  $dF$  has a first differential at  $x_0$ , we can apply the second condition in the definition of first differential, and Theorem 2 to show that  $\|H(a) - d^2F(x_0, d_1x, d_2x)\|$  approaches zero with  $a$ . Since  $H(a)$  is symmetric in the arguments  $d_1x, d_2x$ , so is its limit  $d^2F$ .

To complete the proof by induction, we suppose that the proposition is true for  $n=m$ , and that  $F$  has an  $(m+1)$ st differential at  $x_0$ . Then  $F$  has an  $(m-1)$ st differential  $d^{m-1}F(x, d_1x, \dots, d_{m-1}x)$ , and an  $m$ th differential  $d^mF(x, d_1x, \dots, d_mx)$ , which are continuous on a neighborhood of  $x_0$ . The differential  $d^mF$  is symmetric in the differentials  $d_1x, \dots, d_mx$ , and hence  $d^{m+1}F(x_0, d_1x, \dots, d_{m+1}x)$  is also symmetric in the first  $m$  differentials. Since  $d^{m+1}F$  is the second differential of  $d^{m-1}F$ ,  $d^{m+1}F$  is symmetric in its last two arguments. Hence it is symmetric in all its differential arguments, and the induction is complete.

**Difference functions.** Bliss, Barnett, and Lamson have made some use of the notion of difference function.\* Taylor's theorem shows that a function  $F$  having continuous differentials up to order  $n$  has also continuous difference functions up to order  $n$ . The converse is true only for  $n=1$ . Other relations between different definitions of differentiability in abstract spaces are easily derived.

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\* Cf. Bliss, these Transactions, vol. 21 (1920), p. 79; Barnett, American Journal of Mathematics, vol. 44 (1922), p. 172; Lamson, the same Journal, vol. 42 (1920), p. 243.

HARVARD UNIVERSITY,  
CAMBRIDGE, MASS.

## ALTERNATIVES TO ZERMELO'S ASSUMPTION\*

BY  
ALONZO CHURCH

1. **The axiom of choice.** The object of this paper is to consider the possibility of setting up a logic in which the axiom of choice is false. The way of approach is through the second ordinal class, in connection with which there appear certain alternatives to the axiom of choice. But these alternatives have consequences not only with regard to the second ordinal class but also with regard to other classes, whose definitions do not involve the second ordinal class, in particular with regard to the continuum. And therefore it is possible to consider these alternatives as, in some sense, postulates of logic. In what follows we proceed, after certain introductory considerations, to state these postulates, to inquire into their character, and to derive as many as possible of their consequences.

The axiom of choice, which is also known as Zermelo's assumption,<sup>†</sup> and, in a weakened form, as the multiplicative axiom,<sup>‡</sup> is a postulate of logic which may be stated in the following way:

*Given any set  $X$  of classes which does not contain the null class, there exists a one-valued function,  $F$ , such that if  $x$  is any class of the set  $X$  then  $F(x)$  is a member of the class  $x$ .*

An equivalent statement is that there exists an assignment to every class  $x$  belonging to the set  $X$  of a unique element  $\rho$  such that  $\rho$  is contained in  $x$ .

The important case is that in which the set  $X$  contains an infinite number of classes, because the assertion of the postulate is obviously capable of proof when the number of classes is finite. Accordingly a convenient, although not quite precise, characterization of the axiom of choice is obtained by saying that it is a postulate which justifies the employment of an infinite number of acts of arbitrary choice.

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† E. Zermelo. *Beweis dass jede Menge wohlgeordnet werden kann*, Mathematische Annalen, vol. 59 (1904), p. 514. See also *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Mathematische Annalen, vol. 65 (1908), p. 110, and *Untersuchungen über die Grundlagen der Mengenlehre*, Mathematische Annalen, vol. 65 (1908), p. 266, where Zermelo states the weaker form of the axiom of choice which Russell has called the multiplicative axiom.

‡ B. Russell, *On some difficulties in the theory of transfinite numbers and order types*, Proceedings of the London Mathematical Society, (2), vol. 4 (1907), p. 48; Whitehead and Russell, *Principia Mathematica*, vol. I, 1910, p. 561.

Instead of assuming that the function  $F$  exists in the case of every set  $X$  of classes, it is possible to assume only that the function  $F$  exists if the set  $X$  contains a denumerable infinity of classes.\* Or we may assume that the function  $F$  exists if the set  $X$  contains either  $\aleph_1$  classes or some less number of classes. In this way we obtain a sequence of postulates, each stronger than those which precede it, all of them weakened forms of Zermelo's assumption, which we may call, respectively, the axiom of choice for sets of  $\aleph_0$  classes, the axiom of choice for sets of  $\aleph_1$  classes, and so on.

For our present purpose we wish to exclude all forms of the axiom of choice from among the postulates of logic, so that in what follows no appeal to the axiom of choice is to be allowed.

2. **The second ordinal class.** As defined by Cantor,† the second ordinal class consists of all those ordinals  $\alpha$  such that a well-ordered sequence of ordinal number  $\alpha$  has  $\aleph_0$  as its cardinal number. Instead of this definition we prefer a definition in terms of order alone such as that given in the next paragraph. This definition probably cannot be proved equivalent to Cantor's except with the aid of the axiom of choice for sets of  $\aleph_0$  classes. The relation between the two definitions will appear more clearly in §§ 8, 9, and 10 below.

We shall, therefore, define the second ordinal class by means of the following set of postulates:‡

1. *The second ordinal class is a simply ordered aggregate.*
2. *There is a first ordinal  $\omega$  in the second ordinal class.*
3. *If  $\alpha$  is any ordinal of the second ordinal class, there is a first ordinal,  $\alpha+1$ , of the set of ordinals of the second ordinal class which follow  $\alpha$ .*
4. *If the ordinals  $\beta_0, \beta_1, \beta_2, \dots$  of the second ordinal class are all distinct and form, in their natural order, an ordered sequence ordinally similar to the sequence  $0, 1, 2, 3, \dots$  of positive integers, there is an ordinal  $\beta$  of the second ordinal class, the upper limit of the sequence  $\beta_0, \beta_1, \beta_2, \dots$ , which is the first ordinal in the set of ordinals which follow every ordinal  $\beta_i$  of this sequence.*
5. *There is no proper subset of the second ordinal class which contains the ordinal  $\omega$  and which has the property that if it contains the ordinal  $\alpha$  it contains also  $\alpha+1$ , and if it contains a sequence  $\beta_0, \beta_1, \beta_2, \dots$  of the kind described in Postulate 4 it contains also the upper limit  $\beta$ .*

\* Cf. B. Russell, *Introduction to Mathematical Philosophy*, 1919, p. 129.

† G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, *Mathematische Annalen*, vol. 49 (1897), p. 227.

‡ A closely similar definition of the second ordinal class has been given by O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, these *Transactions*, vol. 6 (1905), p. 170.

The fifth postulate makes possible the process of transfinite induction.

The positive integers, including 0, are thought of as forming the first ordinal class and as preceding the ordinal  $\omega$  in their natural order, so that  $\omega$  is the upper limit of the sequence 0, 1, 2, 3, . . . .

The ordinal  $\Omega$  is the first ordinal which follows all the ordinals of the second ordinal class. It belongs to the third ordinal class, or would belong to this class if we chose to construct it, as could be done by means of a set of postulates analogous to Postulates 1-5.

If an ordinal  $\alpha$  precedes an ordinal  $\beta$  in the arrangement of the ordinals just described (which we shall call the natural order of the ordinals), we say that  $\alpha$  is less than  $\beta$  and  $\beta$  is greater than  $\alpha$ .

An ordinal  $\beta$ , other than 0, of the first or the second ordinal class is of the first kind or of the second kind according as there is or is not a greatest ordinal less than  $\beta$ .

The *upper limit*  $\alpha$  of a sequence  $s$  of ordinals in their natural order such that  $s$  contains no greatest ordinal is the least ordinal greater than all the ordinals in  $s$ . This ordinal  $\alpha$  always exists if  $s$  contains no greatest ordinal, but  $\alpha$  may sometimes belong to a higher ordinal class than any ordinal in  $s$ .

A sequence of distinct ordinals of the first and second ordinal classes, in their natural order,  $\beta_0, \beta_1, \beta_2, \dots$ , ordinally similar to the sequence 0, 1, 2, 3, . . . of positive integers, is said to be a *fundamental sequence* of its upper limit  $\beta$ .

A sequence  $t$  of ordinals in their natural order is *internally closed* if it contains the upper limits of all its sub-sequences which have an upper limit different from the upper limit of  $t$ .

We shall not prove explicitly as consequences of Postulates 1-5 all the theorems about the second ordinal class which we shall need, but we shall make use freely of known theorems whenever these theorems do not depend on the axiom of choice.

The set of all ordinals which are less than a given ordinal  $\alpha$  forms, when these ordinals are arranged in their natural order, a well-ordered sequence, which is called the *segment* determined by  $\alpha$ , and  $\alpha$  is said to be the *ordinal number* of this sequence and of all well-ordered sequences ordinally similar to it. In particular,  $\omega$  is the ordinal number of the sequence of positive integers in their natural order.

The notions of addition, multiplication, and exponentiation of ordinals, of which we shall need to make some use, either may be defined\* in terms of

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\*G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, *Mathematische Annalen*, vol. 49 (1897), pp. 207-218, and pp. 231-235.

the notion of the ordinal number of a well-ordered sequence or may be defined more directly from the postulates by means of a process of induction.

3. **Notations for cardinal numbers.** The cardinal number of the segment determined by an ordinal  $\alpha$  is called the cardinal number corresponding to  $\alpha$ . Those cardinals which correspond to some ordinal greater than or equal to  $\omega$  are called aleph cardinals. When arranged in order of magnitude they form a well ordered sequence. The first of them is the cardinal number corresponding to  $\omega$ , which we call\*  $\aleph_0$ . The remainder of them are denoted by the letter  $\aleph$  (aleph) with an ordinal as a subscript, this ordinal indicating the position of the number in the well-ordered sequence of aleph cardinals.

The cardinal number corresponding to  $\Omega$  is different† from  $\aleph_0$ . The question whether or not it is  $\aleph_1$ , the first aleph cardinal after  $\aleph_0$ , is left open for discussion below.

If the cardinal number of an aggregate  $S$  is  $\aleph_\alpha$ , the cardinal number  $\beth_\alpha$  of the aggregate of subsets of  $S$  is greater than  $\aleph_\alpha$ . These cardinal numbers we call beth cardinals and denote them by the letter  $\beth$  (beth) with the same subscript as the corresponding aleph. Besides these we may define a set of beth cardinals with two subscripts as follows. If the cardinal number of an aggregate  $S$  is  $\aleph_\alpha$ , and  $\aleph_\beta$  is an aleph cardinal less than  $\aleph_\alpha$ , then  $\beth_{\alpha,\beta}$  is the cardinal number of the aggregate of those subsets of  $S$  which have a cardinal number not greater than‡  $\aleph_\beta$ .

The question of the distinctness of these cardinals from the aleph cardinals and from one another must be left open.

The cardinal number  $\beth_0$  is, by definition, the cardinal number of the class of all classes of positive integers. It is the cardinal number of the continuum of real numbers. § It is also the cardinal number of the set of real numbers on a segment of the continuum, the cardinal number of the set of irrational numbers of the continuum, and the cardinal number of the set of irrational

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\* G. Cantor *Beiträge zur Begründung der transfiniten Mengenlehre*, erster Artikel, *Mathematische Annalen*, vol. 46 (1895), p. 492.

† G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, zweiter Artikel, *Mathematische Annalen*, vol. 49 (1897), p. 227.

‡ The cardinal numbers defined in this paragraph are all infinite both in the sense of being non-inductive and in the sense of being reflexive. We shall not be concerned with non-inductive non-reflexive cardinals, although the existence of these cardinals seems to be possible if we deny the axiom of choice. On this class of cardinals see Whitehead and Russell, *Principia Mathematica*, vol. II, 1912, p. 278 and p. 288.

§ G. Cantor, loc. cit., erster Artikel, p. 488. The class of subsets of a class of cardinal number  $b$  evidently has the cardinal number  $2^b$  as defined by Cantor.

numbers on a segment of the continuum. And by means of the expansion of an irrational number as a continued fraction it can be shown that  $\aleph_0$  is the cardinal number of the class of sequences of positive integers of ordinal number  $\omega$ .

4. **The cardinal number of the class of all well-ordered rearrangements of the positive integers.** By a well-ordered rearrangement of the positive integers is to be understood a well-ordered sequence of positive integers such that in it every positive integer occurs once and but once. The well-ordered sequence may be of any possible ordinal number, but no positive integer may be repeated in the sequence and none may be omitted from it. With this understanding we shall prove the following theorem:

**THEOREM 1.** *The class of all well-ordered rearrangements of the positive integers has the cardinal number  $\aleph_0$  of the continuum.*

For the class of all ordered pairs of positive integers is of cardinal number\*  $\aleph_0$ . Therefore the set  $P$  of all classes of ordered pairs of positive integers is of cardinal number  $\aleph_0$ .

Now to every well-ordered rearrangement  $W$  of the positive integers corresponds a class  $Q$  of ordered pairs of positive integers such that the ordered pair  $(a, b)$  is contained in  $Q$  if and only if  $a$  precedes  $b$  in  $W$ . And no such class  $Q$  of ordered pairs of positive integers corresponds in this way to more than one well-ordered rearrangement of the positive integers. Therefore the class of all well-ordered rearrangements of the positive integers can be put into one-to-one correspondence with a part of the set  $P$  and therefore with a part of the continuum (because  $P$  can be put into one-to-one correspondence with the continuum).

But the class of all well-ordered rearrangements of the positive integers contains a part which can be put into one-to-one correspondence with the continuum, namely the class  $O$  of those well-ordered rearrangements of ordinal number  $\omega$  which have the property that in them the set of odd positive integers and the set of even positive integers occur each in its natural order (so that  $O$  contains, for example, the sequence 0, 2, 1, 4, 6, 3, 8, 10, 5, 12, 14, 7,  $\dots$ ). For to the well-ordered rearrangement  $a_0, a_1, a_2, \dots$  contained in  $O$  can be correlated the irrational number  $(b_0/2) + (b_1/2^2) + (b_2/2^3) + \dots$  where  $b_i$  is equal to 0 or 1 according as  $a_i$  is even or odd. In this way can be set up a one-to-one correspondence between  $O$  and the set of irrational numbers between 0 and 1 and therefore between  $O$  and the continuum.

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\*G. Cantor, loc. cit., erster Artikel, p. 494.

The theorem to be proved, therefore, follows by an appeal to the theorem of Schröder and Bernstein\* which states that if each of two classes can be put into one-to-one correspondence with a part of the other, then the two classes can be put into one-to-one correspondence.†

**COROLLARY 1.** *The continuum can be divided into  $\aleph_1$  mutually exclusive subsets, each of cardinal number  $\beth_0$ .*

For the class of well-ordered rearrangements of the positive integers can be so divided by classifying the well-ordered rearrangements of the positive integers according to their ordinal number.

**COROLLARY 2.** *The class of all well-ordered sequences of positive integers which contain no repetition of any integer is of cardinal number  $\beth_0$ .*

This corollary and the two following can be proved by the same argument as that used in proving Theorem 1.

**COROLLARY 3.** *The class of all permutations of the positive integers is of cardinal number  $\beth_0$ , where a permutation of the positive integers is restricted to be of ordinal number  $\omega$ .*

\* E. Schröder, *Über zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze*, Nova Acta Academiae Caesareae Leopoldino-Carolinae Germanicae Naturae Curiosorum, vol. 71 (1898), pp. 336-340; E. Borel, *Leçons sur la Théorie des Fonctions*, pp. 102-107; A. Korselt, *Über einen Beweis des Äquivalenzsatzes*, Mathematische Annalen, vol. 70 (1911), pp. 294-296.

A very neat proof is given by J. König, *Sur la théorie des ensembles*, Comptes Rendus, vol. 143 (1906), pp. 110-112.

† The Schröder-Bernstein theorem can be proved in such a way that an explicit one-to-one correspondence between the classes in question is set up. Therefore we are able here actually to set up an explicit one-to-one correspondence between the continuum and the class of well-ordered rearrangements of the positive integers. And this means, of course, that the sets of Corollary 1 are explicitly defined subsets of the continuum.

When we have made explicit in this way the one-to-one correspondence between the continuum and the class of well-ordered rearrangements of the positive integers, it is not difficult to say in simple cases what well-ordered rearrangement of the positive integers corresponds to a given number  $\alpha$  of the continuum. But in certain cases the answer to this question involves the solving of difficult (conceivably unsolvable) problems about the dual fractional expansion of  $\alpha$  similar to that proposed by L. E. J. Brouwer, Mathematische Annalen, vol. 83 (1921), pp. 209-210, for the decimal expansion  $\pi$ , but more complicated in character. But the correspondence which we have set up is none the less explicit.

‡ W. Sierpinski, in Bulletin International de l'Académie des Sciences de Cracovie, 1918, p. 110, gives another proof, independent of the axiom of choice, that the continuum can be divided into  $\aleph_1$  mutually exclusive subsets. The division of the continuum which he effects is actually a division into subsets each of cardinal number  $\beth_0$ , although he does not prove this.

§ F. Bernstein, *Untersuchungen aus der Mengenlehre*, Mathematische Annalen, vol. 61 (1905), p. 142.

COROLLARY 4. *The class of all simply ordered sequences of positive integers which contain no repetition of any integer is of cardinal number\*  $\aleph_0$ .*

The preceding theorem and corollaries are independent of the axiom of choice.

5. **The categorical character of the set of postulates 1-5.** Between any two aggregates  $J$  and  $J'$  both of which satisfy Postulates 1-5 of §2 it is possible to set up in the following way a one-to-one correspondence which preserves order. Let the first element  $\omega$  of  $J$  correspond to the first element  $\omega'$  of  $J'$ . Then make the requirement that if the element  $\alpha$  of  $J$  correspond to the element  $\alpha'$  of  $J'$  then the element  $\alpha+1$  of  $J$  shall correspond to the element  $\alpha'+1$  of  $J'$  and to no other element of  $J'$ , and that if the elements of a fundamental sequence  $\beta_0, \beta_1, \beta_2, \dots$  in  $J$  correspond respectively to the elements of a fundamental sequence  $\beta'_0, \beta'_1, \beta'_2, \dots$  in  $J'$  then the upper limit  $\beta$  of the first sequence shall correspond to the upper limit  $\beta'$  of the second sequence and to no other element of  $J'$ .

Now no element of either aggregate follows next after more than one element of the aggregate. And in either aggregate two fundamental sequences have the same upper limit if and only if it is true that any given ordinal of either sequence precedes some ordinal of the other sequence. From this it follows that if a correspondence between  $J$  and  $J'$  which satisfies the requirement just stated is one-to-one and preserves order in the case of every element which precedes a given element  $\alpha$  of  $J$ , then it is one-to-one and preserves order in the case of  $\alpha$  also. Therefore we can establish by transfinite induction the existence of a one-to-one correspondence between  $J$  and  $J'$  which preserves order, the correspondence being constructed in a step by step fashion in accordance with the requirement of the preceding paragraph.

If  $\alpha$  is an element of the second kind in  $J$ , the corresponding element  $\alpha'$  in  $J'$  is obtained by choosing a fundamental sequence  $\alpha_0, \alpha_1, \alpha_2, \dots$  for  $\alpha$ . Then if  $\alpha'_0, \alpha'_1, \alpha'_2, \dots$  in  $J'$  correspond respectively to  $\alpha_0, \alpha_1, \alpha_2, \dots$  in  $J$ ,  $\alpha'$  is the upper limit of the fundamental sequence  $\alpha'_0, \alpha'_1, \alpha'_2, \dots$  in  $J'$ . It is true, however, that no matter what fundamental sequence is chosen for  $\alpha$  the same corresponding element  $\alpha'$  in  $J'$  is obtained. The construction of the one-to-one correspondence between  $J$  and  $J'$  involves, accordingly, no appeal to the axiom of choice.

\* Cf. F. Bernstein, *Untersuchungen aus der Mengenlehre*, Mathematische Annalen, vol. 61 (1905), pp. 140-145, where it is proved that  $\aleph_0$  is the cardinal number of the class of all order types in which the set of positive integers can be arranged. Two proofs of this theorem are given, but in both of them it is necessary to use the axiom of choice. Nevertheless an obvious modification of the first of these proofs suffices to prove without the aid of the axiom of choice, not the theorem stated by Bernstein, but the related theorem stated in Corollary 4 above.

It will be convenient in this connection to use the following definition of categorical character. A set of postulates is categorical if, given any two systems both of which satisfy all the postulates of the set, there exists a one-to-one correspondence between these two systems which preserves all the relations among their elements which appear as undefined terms in the postulates.

The only such relation which appears in the set of postulates 1-5 under discussion is the relation of order among the ordinals. Therefore we have proved that this set is categorical in the sense just defined.

6. **Nature of a categorical set.** Suppose that a categorical set  $S$  of postulates is given, and two contradictory statements  $K$  and  $L$  in the form of theorems about the system described by  $S$ . Then we are not at liberty to suppose that there exist two systems  $s_1$  and  $s_2$ , both of which satisfy  $S$ , in one of which  $K$  is true, and in the other of which  $L$  is true, because, if this were the case, we could obtain a contradiction at once by means of the one-to-one correspondence between  $s_1$  and  $s_2$ . In a certain sense, therefore, a categorical set is a complete set, because it is impossible to employ simultaneously two distinct systems which satisfy the same categorical set of postulates.

It does not, however, follow that one of the statements  $K$  or  $L$  must be inconsistent with the set of postulates\*  $S$ . It is quite conceivable that, although the coexistence of  $s_1$  and  $s_2$  lead to contradiction, nevertheless neither the existence of  $s_1$  alone nor that of  $s_2$  alone should lead to contradiction.

It is clear that the completeness of the set of postulates at the basis of our logic is involved.† We might, not unnaturally, make it one of the requirements for completeness of a set of postulates for logic that, in all such cases as that described above, one of the statements  $K$  or  $L$  should lead to a contradiction when taken in conjunction with the set  $S$ . In the absence,

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\* E. B. Wilson, *Logic and the continuum*, Bulletin of the American Mathematical Society, vol. 14 (1908), pp. 432-443. See also E. V. Huntington, *A set of postulates for ordinary complex algebra*, these Transactions, vol. 6 (1905), p. 210 and the second footnote on that page.

† It is possible to demonstrate the completeness of a certain portion of the postulates of logic in the sense that no new independent and consistent postulate can be added to this portion without introducing a new undefined term. The portion in question consists of the postulates given by Whitehead and Russell in part I, section A, of the *Principia Mathematica*. The proof of the completeness of these postulates has been given by E. L. Post, *Introduction to a general theory of elementary propositions*, American Journal of Mathematics, vol. 43 (1921), p. 163-185. But this does not imply the completeness in any sense of the full set of postulates for logic (as at present known), because this full set involves additional undefined terms.

however, of any demonstration that the set of postulates on which our logic is based satisfies this requirement, we must not infer from the fact that a set of postulates  $S$  is categorical that there do not exist one or more independent postulates which can be added to the set.

It is not improbable that the set of postulates at the basis of our logic is not complete, even if the axiom of choice is included in the set, because if it were complete it ought to be possible, in the case of every set  $X$  of classes which does not contain the null class, to construct a particular function  $F$  of the kind whose existence is required by the axiom of choice, a construction the possibility of which is, in many cases, doubtful. If the axiom of choice is excluded the probability of completeness is even more remote.

The question whether the set of postulates at the basis of our logic is or is not complete is evidently equivalent to the question whether or not every mathematical problem can be solved.\* On the other hand, since each of these questions is in the form of a theorem about the postulates of logic, into the truth of which theorem it is proposed to inquire, neither has a direct connection with the law of excluded middle,† which is itself a postulate of logic. Suppose, for example, that we have before us a certain consistent set  $W$  of postulates for logic among which is the law of excluded middle. There may be, if  $W$  is not complete, a postulate  $p$  such that either  $p$  or  $\text{not-}p$  can be added to the set  $W$  without destroying the consistent character of the set. In this case there may be a universe of discourse  $U_1$  in which  $p$  and the postulates of  $W$  are satisfied and also a universe of discourse  $U_2$  in which  $\text{not-}p$  and the postulates of  $W$  are satisfied. Then  $p$  would satisfy the law of excluded middle both in  $U_1$  and in  $U_2$ , in  $U_1$  by being true, and in  $U_2$  by being false. Accordingly our inability to conclude on the basis of  $W$  whether  $p$  is true or false does not prevent our concluding on the basis of  $W$  that  $p$  is either true or false.

7. **Alternatives to Zermelo's assumption.** The foregoing discussion is intended to prepare the way for the suggestion that there may be one or more additional independent postulates which can be added to the set of postulates 1-5 and to forestall the objection that this set is already categorical.

With this possibility in mind we propose to examine the consequences of each of the following postulates when it is taken in conjunction with Postulates 1-5:

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\* The latter question is proposed by D. Hilbert in *Mathematische Probleme*, Göttinger Nachrichten, 1900, pp. 261-262, and *Archiv der Mathematik und Physik*, (3), vol. 1 (1901), p. 52, and *Mathematical Problems*, Bulletin of the American Mathematical Society, vol. 8 (1902), pp. 444-445.

† The opposite view is maintained by L. E. J. Brouwer. See for example, *Intuitionistische Mengenlehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 28 (1920), pp. 203-208.

A. *There exists an assignment of a unique fundamental sequence to every ordinal of the second kind in the second ordinal class.*

B. *There exists no assignment of a unique fundamental sequence to every ordinal of the second kind in the second ordinal class; but given any ordinal  $\alpha$  of the second ordinal class, there exists an assignment of a unique fundamental sequence to every ordinal of the second kind less than  $\alpha$ .*

C. *There is an ordinal  $\phi$  of the second ordinal class such that there exists no assignment of a unique fundamental sequence to every ordinal of the second kind less than  $\phi$ .*

It is an immediate consequence of Postulates 1-5 that there exist fundamental sequences for any particular ordinal of the second kind in the second ordinal class. The preceding postulates A, B, and C are concerned with the possibility of assigning a particular fundamental sequence to every such ordinal in a simultaneous manner.\*

Postulate A can be derived as a consequence of the axiom of choice for sets of  $\aleph_1$  classes. Postulate B implies a denial of the axiom of choice for sets of  $\aleph_1$  classes but seems to be consistent with this axiom for sets of  $\aleph_0$  classes. Postulate C implies a denial of the axiom of choice for sets of  $\aleph_0$  classes. There is no reason, however, to suppose that Postulate B implies the axiom of choice for sets of  $\aleph_0$  classes or that Postulate A implies this axiom for sets of  $\aleph_1$  classes.

Postulates A, B, and C are mutually exclusive and it is clear that, together, they exhaust the conceivable alternatives. There are, therefore, three conceivable kinds of second ordinal classes, one corresponding to each of these postulates. If any one of these involve a contradiction it is reasonable to expect that a systematic examination of its properties will ultimately reveal this contradiction. But if a considerable body of theory can be developed on the basis of one of these postulates without obtaining inconsistent results, then this body of theory, when developed, could be used as presumptive evidence that no contradiction existed.

If there be two of these postulates neither of which leads to contradiction, then there are corresponding to them two distinct self-consistent second ordinal classes, just as euclidean geometry and Lobachevskian geometry are distinct self-consistent geometries, with, however, this difference, that the two

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\*For a discussion of the problem of carrying out such an assignment of fundamental sequences see O. Veblen, *Continuous increasing functions of finite and transfinite ordinals*, these Transactions, vol. 9 (1908), pp. 280-292.

second ordinal classes are incapable of existing together in the same universe of discourse.

It is not unlikely that no one of the three postulates A, B, C leads to any contradiction.

8. **Consequences of Postulate A.** THEOREM A<sub>1</sub>. *If  $\alpha$  is any ordinal of the second ordinal class, the cardinal number corresponding to  $\alpha$  is\*  $\aleph_0$ .*

For assign to every ordinal  $\beta$  of the second kind which is less than or equal to  $\alpha$  a fundamental sequence  $u_\beta$  (Postulate A).

The ordinals which precede  $\omega$  form, when arranged in their natural order, a sequence of ordinal number  $\omega$ . With this as a starting point, we assign to the ordinals which follow  $\omega$ , one by one in order, an arrangement of all preceding ordinals in a sequence of ordinal number  $\omega$ , in the following way.

When we have assigned an arrangement in a sequence  $l$  of ordinal number  $\omega$  of all ordinals which are less than an ordinal  $\gamma$ , an arrangement in a sequence of ordinal number  $\omega$  of all ordinals which are less than  $\gamma+1$  is obtained by placing  $\gamma$  before  $l$ .

When, to every ordinal  $\zeta$  which is less than an ordinal  $\beta$  of the second kind, we have assigned an arrangement in a sequence  $l_\zeta$  of ordinal number  $\omega$  of all ordinals which are less than  $\zeta$ , the sequences  $l_{\beta_0}, l_{\beta_1}, l_{\beta_2}, \dots$ , where  $\beta_0, \beta_1, \beta_2, \dots$  is the fundamental sequence  $u_\beta$ , may be written one below the other so as to obtain the following array:

$$\begin{array}{ccccccc} \delta_{00}, & \delta_{01}, & \delta_{02}, & \dots & & & \\ \delta_{10}, & \delta_{11}, & \delta_{12}, & \dots & & & \\ \delta_{20}, & \delta_{21}, & \delta_{22}, & \dots & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

where  $\delta_{i0}, \delta_{i1}, \delta_{i2}, \dots$  is the sequence  $l_{\beta_i}$  and contains all the ordinals which are less than  $\beta_i$ . Then the ordinals  $\delta_{ij}$  may be arranged in a sequence of ordinal number  $\omega$  as follows:

$$\delta_{00}, \delta_{01}, \delta_{10}, \delta_{01}, \delta_{11}, \delta_{20}, \delta_{03}, \delta_{12}, \delta_{21}, \delta_{30}, \dots$$

By omitting from this sequence all occurrences of any ordinal after the first occurrence, so that a sequence without repetitions results, we obtain an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\beta$ .

\*See G. Cantor, loc. cit., zweiter Artikel, p. 221. As already explained, Cantor uses the property described in Theorem A<sub>1</sub> in defining the second ordinal class. He then proves, with the aid of the axiom of choice, that the second ordinal class has the properties expressed by the postulates of §2 above.

The way in which the axiom of choice is involved is pointed out by Whitehead and Russell, *Principia Mathematica*, vol. III, 1913, p. 170.

We may prove by induction that this process continues until we obtain an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\alpha$ . Therefore the cardinal number corresponding to  $\alpha$  is  $\aleph_0$ .

**COROLLARY 1.** *There exists an assignment to every ordinal  $\beta$  in the second ordinal class of an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\beta$ .*

For, under Postulate A, we may assign a fundamental sequence  $u_\beta$  to every ordinal of the second kind in the second ordinal class. The process just described then continues until we have assigned to every ordinal  $\beta$  of the second ordinal class an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\beta$ .

**COROLLARY 2.** *There exists an assignment to every ordinal  $\beta$  in the second ordinal class of a well-ordered rearrangement of the positive integers of ordinal number  $\beta$ .*

Because an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\beta$  determines a one-to-one correspondence between the ordinals less than  $\beta$  and the positive integers, and this one-to-one correspondence determines in turn a well-ordered rearrangement of the positive integers of ordinal number  $\beta$ .

**THEOREM A<sub>2</sub>.** *If the class  $R$  consists of all well-ordered sequences of positive integers (allowing any number of repetitions of the same integer) whose ordinal numbers belong to the second ordinal class, the cardinal number of  $R$  is  $\beth_{1,0}$ .*

Let  $O$  be the class of ordinals less than  $\Omega$ . Then the cardinal number of  $O$  is  $\aleph_1$ . Let  $Q$  be the class of the subsets of  $O$  which have a cardinal number not greater than  $\aleph_0$ . Then the cardinal number of  $Q$  is  $\beth_{1,0}$ .

The class  $Q$  can be put into one-to-one correspondence with a part of  $R$  as follows. To the subset  $S$  of  $O$  contained in  $Q$  correlate the sequence  $a_0, a_1, a_2, \dots, a_\omega, \dots$  of  $R$ , where  $a_\mu$  is 1 or 0 according as  $\mu$  is or is not contained in  $S$ , and the sequence consists of those  $a$ 's whose subscripts are less than  $\beta$ , where  $\beta$  is the least ordinal greater than every ordinal in  $S$  and not less than  $\omega$ .

The class  $R$  can be put into one-to-one correspondence with a part of  $Q$  as follows. To the sequence  $b_0, b_1, b_2, \dots, b_\omega, \dots$  of  $R$ , of ordinal number  $\gamma$ , correlate the subset of  $O$  consisting of the following ordinals:  $0, 1, 2, \dots, b_0; \omega, \omega+1, \dots, \omega+b_1; 2\omega, 2\omega+1, \dots, 2\omega+b_2; \dots, \gamma\omega$ .\*

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\*We adopt Cantor's earlier notation, placing the multiplier before the multiplicand.

Therefore, by the Schröder-Bernstein theorem,\*  $Q$  and  $R$  can be put into one-to-one correspondence.

Therefore the cardinal number of  $R$  is  $\beth_{1,0}$ .

THEOREM A<sub>3</sub>. *The cardinal numbers  $\beth_0$  and  $\beth_{1,0}$  are identical.*

In accordance with Corollary 1 of Theorem A<sub>1</sub>, assign to every ordinal  $\beta$  of the second ordinal class an arrangement in a sequence  $t_\beta$  of ordinal number  $\omega$  of all ordinals less than  $\beta$ .

Let  $b$  be one of the sequences belonging to the class  $R$  of the preceding theorem, and let  $\beta$  be the ordinal number of  $b$ . Then, corresponding to  $b$ , we can determine a sequence  $c$  of positive integers of ordinal number  $\omega$  by the rule that if  $\kappa_i$  is the  $i$ th ordinal of  $t_\beta$  then the  $i$ th positive integer in  $c$  shall be the  $\kappa_i$ th positive integer of  $b$ . In this way we can set up a one-to-one correspondence between all the sequences  $b$  of  $R$  which have a fixed ordinal number  $\beta$  and the class of sequences of positive integers of ordinal number  $\omega$ . And in exactly the same way we can set up a one-to-one correspondence between those well-ordered rearrangements of the positive integers which have a fixed ordinal number  $\beta$  and the class of permutations of the positive integers, where a permutation of the positive integers is restricted to be of ordinal number  $\omega$ . But the class of sequences of positive integers of ordinal number  $\omega$  and the class of permutations of the positive integers can be put into one-to-one correspondence, since each is of cardinal number  $\beth_0$ . Therefore, choosing a particular such one-to-one correspondence  $C$ , we can set up a one-to-one correspondence  $K_\beta$  between the sequences of  $R$  which have a fixed ordinal number  $\beta$  and the well-ordered rearrangements of the positive integers which have the ordinal number  $\beta$ . Moreover, since  $C$  can be chosen once for all, we have a uniform method of setting up the one-to-one correspondences  $K_\beta$ , and therefore, without appeal to the axiom of choice, we can suppose them all set up, for every ordinal  $\beta$  of the second ordinal class. But as soon as this is done we have a one-to-one correspondence between  $R$  and the class of well-ordered rearrangements of the positive integers. And, in Theorems A<sub>2</sub> and 1, we have shown that these two classes have, respectively, the cardinal number  $\beth_{1,0}$  and the cardinal number  $\beth_0$ . Therefore these two cardinal numbers are identical.

COROLLARY. *The class  $R$  of Theorem A<sub>2</sub> can be put into one-to-one correspondence with the continuum.*

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\* Loc. cit.

THEOREM A<sub>4</sub>. *The continuum contains a subset of cardinal number\*  $\aleph_1$ .*

For the class  $R$ , which has just been shown to have the cardinal number of the continuum, contains such a subset, namely the set of all those sequences belonging to  $R$  which consist entirely of 2's.

The same theorem can be proved by means of Corollary 2 of Theorem A<sub>1</sub>, because, in accordance with that corollary, the class of well-ordered rearrangements of the positive integers contains a subset of cardinal number  $\aleph_1$ , and, by Theorem 1, this class can be put into one-to-one correspondence with the continuum.

The fact that the set of postulates 1-5 and A, which are all statements about the second ordinal class, has consequences about an entirely different aggregate, namely the continuum, is evidently connected with the fact that these postulates contain more than is necessary to render them categorical.

We have already observed that if we are to think of the three second ordinal classes, that corresponding to Postulate A, that corresponding to B, and that corresponding to C, as all existing we must think of them as each existing in a different universe of discourse. Therefore when we single out one of these second ordinal classes for consideration we thereby restrict the character of the universe of discourse within which we are working. In this way we may think of Postulate A as being indirectly a postulate of logic, although it is in form a statement about the second ordinal class. And the same remark applies to Postulates B and C.

9. **Consequences of Postulate B.** THEOREM B<sub>1</sub>. *If  $\alpha$  is any ordinal of the second ordinal class, the cardinal number corresponding to  $\alpha$  is  $\aleph_0$ .*

The proof of Theorem A<sub>1</sub> applies without change.

THEOREM B<sub>2</sub>. *There exists no assignment to every ordinal  $\beta$  in the second ordinal class of an arrangement in a sequence of ordinal number  $\omega$  of all ordinals less than  $\beta$ .*

For, given an arrangement in a sequence  $t$ , of ordinal number  $\omega$ , of all ordinals less than  $\beta$ , we could obtain a fundamental sequence for  $\beta$  by omitting from  $t$  all the ordinals  $\alpha$  which did not have the property that every ordinal which preceded  $\alpha$  in  $t$  also preceded  $\alpha$  in the natural order of the ordinals. Therefore if there existed an assignment to every ordinal  $\beta$  in the second ordinal class of an arrangement in a sequence of ordinal number  $\omega$

\* A proof of this theorem has been given by G. H. Hardy, *A theorem concerning the infinite cardinal numbers*, Quarterly Journal of Pure and Applied Mathematics, vol. 35 (1903), pp. 87-94. The use of a simultaneous assignment of a fundamental sequence to every ordinal of the second kind in the second ordinal class is essential to Hardy's proof, as it is to the proof given above.

of all ordinals less than  $\beta$ , there would exist an assignment of a fundamental sequence to every ordinal  $\beta$  in the second ordinal class, contrary to Postulate B.

**COROLLARY.** *There exists no assignment to every ordinal  $\beta$  in the second ordinal class of a well-ordered rearrangement of the positive integers of ordinal number  $\beta$ .*

**THEOREM B<sub>3</sub>.** *The continuum cannot be well-ordered.*

For it follows from Theorem 1 that, if the continuum could be well-ordered, the set of all well-ordered rearrangements of the positive integers could be arranged in a well-ordered sequence  $u$ . And it would then be possible to assign to every ordinal  $\beta$  in the second ordinal class a well-ordered rearrangement of the positive integers of ordinal number  $\beta$ , because we could choose, for every  $\beta$ , the first rearrangement of ordinal number  $\beta$  that occurred in  $u$ . This, however, is contrary to the corollary of the preceding theorem.

**THEOREM B<sub>4</sub>.** *If the class  $R$  consists of all well-ordered sequences of positive integers (allowing any number of repetitions of the same integer) whose ordinal numbers belong to the second ordinal class, the cardinal number of  $R$  is  $\beth_{1,0}$ .*

The proof of Theorem A<sub>2</sub> applies without change.

**DEFINITION.** Let  $t$  be an internally closed sequence of ordinals of the first and second ordinal classes, all distinct, and arranged in their natural order. Then there is one and only one way in which  $t$  can be put into one-to-one correspondence (preserving order) with the sequence of ordinals less than  $\Omega$  or a segment of this sequence. The ordinals of the first and second ordinal classes which are correlated to themselves in this correspondence form, when arranged in their natural order, an internally closed sequence  $t'$ , the *first derived sequence*\* of  $t$ . The first derived sequence of  $t'$  is the *second derived sequence* of  $t$  and so on. If  $\nu$  is an ordinal of the second kind in the second ordinal class, the  $\nu$ th derived sequence  $t''$  of  $t$  consists of those ordinals which are contained in all previous derived sequences. If  $\mu$  is an ordinal of the first or second ordinal class the  $(\mu+1)$ th derived sequence  $t^{(\mu+1)}$  of  $t$  is the first derived sequence of  $t^{(\mu)}$ .

If  $\mu$  is an ordinal of the first or second class, the  $\mu$ th derived sequence  $t^{(\mu)}$  of  $t$  is, like  $t$ , an internally closed sequence of ordinals of the first and second

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\*The sequence  $t$  determines by its correspondence with the sequence of ordinals less than  $\Omega$  or a segment thereof a continuous increasing function. The sequence  $t'$  consists of the set of values of the first derived function. See O. Veblen, *Continuous increasing functions of finite and transfinite ordinals*, these Transactions, vol. 9 (1908), p. 281.

ordinal classes, all distinct, and arranged in their natural order. And if the ordinal number of  $l$  is  $\Omega$  that of  $l^2$  is also\*  $\Omega$ .

If the first term of  $l$  is greater than 0 the sequence formed by taking the first term of each of the derived sequences of  $l$  in order is an internally closed sequence of distinct ordinals in their natural order.†

DEFINITION. An internally closed sequence  $r$  of ordinals of the second kind of the second ordinal class, all distinct and arranged in their natural order, is a *reduction sequence* if there exists an assignment to every ordinal  $\kappa$  of the second kind in the second ordinal class of a sequence  $v_\kappa$  of distinct ordinals arranged in their natural order, such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is either  $\omega$  or one of the ordinals of  $r$ .

It follows at once from Postulate B that the ordinal number of a reduction sequence must be  $\Omega$ .

The sequence of self-residual‡ ordinals of the second ordinal class in their natural order is a reduction sequence. Its first derived sequence, the sequence of  $\epsilon$ -numbers,§ is also a reduction sequence.

THEOREM B<sub>5</sub>. *If the first derived sequence of a reduction sequence  $r$  is a reduction sequence, then all the successive derived sequences of  $r$  are reduction sequences.*

For let  $r^\nu$  be the  $\nu$ th derived sequence of  $r$ . Assign a fundamental sequence to every ordinal of the second kind less than  $\nu$  (Postulate B). And to every ordinal of the second kind in the second ordinal class assign a sequence  $v_\kappa$  of distinct ordinals arranged in their natural order such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is either  $\omega$  or one of the ordinals of  $r'$ . This is possible since, by hypothesis,  $r'$  is a reduction sequence.

Let  $\rho$  be an ordinal of the second kind which occurs in  $r'$  but not in  $r^\nu$ . Then there is a last ordinal  $\mu$ , which is less than  $\nu$ , such that  $\rho$  occurs in  $r^\mu$ . Let  $\alpha$  be the ordinal which corresponds to  $\rho$  in a one-to-one correspondence (preserving order) between  $r^\mu$  and the sequence of all ordinals less than  $\Omega$ .

If  $\mu$  is of the second kind, let  $\mu_0, \mu_1, \mu_2, \dots$  be the fundamental sequence which we have assigned to  $\mu$ .

If  $\mu$  is of the second kind and  $\alpha$  is equal to 0, so that  $\rho$  stands in the first place in  $r^\mu$ , the ordinals  $\rho_0, \rho_1, \rho_2, \dots$  which stand in the first place in  $r^{\mu_0}, r^{\mu_1}, r^{\mu_2}, \dots$ , respectively, constitute a fundamental sequence for  $\rho$ .

\*O. Veblen, loc. cit., pp. 283-285.

†O. Veblen, loc. cit., p. 285.

‡For definitions see G. Cantor, loc. cit., zweiter Artikel

If  $\mu$  is of the second kind and  $\alpha$  is of the first kind, let  $\sigma$  be the ordinal which next precedes  $\rho$  in  $r^\mu$ , and let  $\rho_i$  be the ordinal which next follows  $\sigma$  in  $r^{\mu_i}$ . The ordinals  $\rho_0, \rho_1, \rho_2, \dots$  are in their natural order, because if  $\rho_{i+1}$  were less than  $\rho_i$  it would follow that  $\rho_{i+1}$  did not occur in  $r^{\mu_i}$ , contrary to the definition of derived sequences, which requires that all the terms of  $r^{\mu_{i+1}}$  shall be ordinals of  $r^{\mu_i}$ . And  $\rho_0, \rho_1, \rho_2, \dots$  are all distinct, because if  $\rho_{i+1}$  were equal to  $\rho_i$  it would follow that both were equal to  $\beta+1$ , where  $\beta$  was the ordinal which corresponded to  $\sigma$  in a one-to-one correspondence (preserving order) between  $r^{\mu_i}$  and the sequence of all ordinals less than  $\Omega$ , contrary to the requirement that all the ordinals of  $r$ , and therefore all those of  $r^{\mu_i}$ , shall be ordinals of the second kind. The upper limit of the sequence  $\rho_0, \rho_1, \rho_2, \dots$ , since it is also the upper limit of the sequence  $\rho_i, \rho_{i+1}, \rho_{i+2}, \dots$ , necessarily occurs in  $r^{\mu_i}$ , and, since it occurs in every  $r^{\mu_i}$ , it occurs also in  $r^\mu$ . This upper limit cannot be greater than  $\rho$ , because, if it were, some  $\rho_i$ , say  $\rho_n$ , would be greater than  $\rho$ , and it would follow that  $\rho$  did not occur in  $r^\mu$ . But the upper limit of  $\rho_0, \rho_1, \rho_2, \dots$  is greater than  $\sigma$ , because each term is greater than  $\sigma$ . Therefore the upper limit is  $\rho$ , so that  $\rho_0, \rho_1, \rho_2, \dots$  is a fundamental sequence for  $\rho$ .

If  $\mu$  is of the first kind (so that  $\mu = \lambda + 1$ ) and  $\alpha$  is equal to 0, let  $\rho_0$  be the first ordinal of  $r^\lambda$ ,  $\rho_1$  the  $\rho_0$ th ordinal of  $r^\lambda$ ,  $\rho_2$  the  $\rho_1$ th ordinal of  $r^\lambda$ , and so on. Then the sequence  $\rho_0, \rho_1, \rho_2, \dots$  is an increasing sequence, because  $\rho_{i+1} < \rho_i$  is impossible on account of the increasing character of  $r^\lambda$ , and  $\rho_{i+1} = \rho_i$  would imply  $\rho_1 = \rho_0$ , a situation which is impossible because  $\rho_0$  cannot be equal to 0 and  $\rho_1$  is the  $\rho_0$ th ordinal of  $r^\lambda$ . And the upper limit of  $\rho_0, \rho_1, \rho_2, \dots$  is  $\rho$ .

If  $\mu$  is of the first kind (so that  $\mu = \lambda + 1$ ) and  $\alpha$  is also of the first kind, let  $\sigma$  be the ordinal which next precedes  $\rho$  in  $r^\mu$ . Let  $\rho_0$  be the  $(\sigma + 1)$ th ordinal of  $r^\lambda$ ,  $\rho_1$  the  $\rho_0$ th ordinal of  $r^\lambda$ ,  $\rho_2$  the  $\rho_1$ th ordinal of  $r^\lambda$ , and so on. Then the sequence  $\rho_0, \rho_1, \rho_2, \dots$  is an increasing sequence, because  $\rho_{i+1} < \rho_i$  is impossible on account of the increasing character of  $r^\lambda$ , and  $\rho_{i+1} = \rho_i$  would imply  $\rho_0 = \sigma + 1$ , whereas  $\rho_0$  must be an ordinal of the second kind. And the upper limit of  $\rho_0, \rho_1, \rho_2, \dots$  is  $\rho$ .

The case  $\mu = 1$  is taken account of in each of the two preceding paragraphs, for in that case  $\lambda = 0$  and  $r^\lambda$  is the sequence  $r$  itself.

If  $\alpha$  is of the second kind, we have assigned to  $\alpha$  a sequence  $v_\alpha$  whose upper limit is  $\alpha$  and whose ordinal number is an ordinal  $\tau$  of  $r'$ . If  $\gamma$  is any ordinal less than  $\tau$ , let  $\alpha_\gamma$  be the  $\gamma$ th term of  $v_\alpha$ , and let  $\rho_\gamma$  be the  $\alpha_\gamma$ th term of  $r^\mu$ . Then the sequence  $\rho_2, \rho_3, \rho_4, \dots, \rho_\omega, \rho_{\omega+1}, \dots$  is a sequence of ordinal number  $\tau$  whose upper limit is  $\rho$ . And  $\alpha$ , and therefore  $\tau$ , is less than  $\rho$ . The problem of finding an increasing sequence whose upper limit is  $\rho$  and whose ordinal number is either  $\omega$  or an ordinal of  $r'$  is, therefore, reduced to the

corresponding problem for the smaller ordinal  $\tau$ , and this reduction continues until we obtain such a sequence or until one of the cases already considered arises.

We can now conclude that it is possible to assign to every ordinal  $\rho$  in  $r'$  an increasing sequence of ordinals whose upper limit is  $\rho$  and whose ordinal number is either  $\omega$  or an ordinal of  $r'$ , because we have just described a systematic method of making such an assignment. But we have assigned to every ordinal  $\kappa$  of the second kind in the second ordinal class an increasing sequence  $v_\kappa$  of ordinals whose upper limit is  $\kappa$  and whose ordinal number is either  $\omega$  or an ordinal of  $r'$ . Therefore we can assign to every ordinal  $\kappa$  of the second kind in the second ordinal class an increasing sequence of ordinals whose upper limit is  $\kappa$  and whose ordinal number is either  $\omega$  or an ordinal of  $r'$ . Therefore  $r''$  is a reduction sequence.

**COROLLARY 1.** *The sequence  $\tilde{r}$  of those ordinals which occur in the first place in the sequences  $r^\theta$ , where  $\theta$  takes on all values which make it an ordinal of the second kind, is a reduction sequence.*

For, by the method just given, we can assign to every ordinal  $\rho$  of  $r'$  an increasing sequence of ordinals whose upper limit is  $\rho$  and whose ordinal number is an ordinal of  $\tilde{r}$ .

If  $\rho$  occurs in one of the sequences  $r^\theta$  in such a way that it has an immediate predecessor  $\sigma$  in  $r^\theta$ , let  $\rho_\alpha$  be the ordinal which next follows  $\sigma$  in  $r^\alpha$ , where  $\alpha$  is any ordinal less than  $\theta$ . Then the ordinals  $\rho_\alpha$  form an increasing sequence of ordinal number  $\theta$  whose upper limit is  $\rho$ . From this, by means of the sequence  $v_\theta$ , we obtain an increasing sequence whose upper limit is  $\rho$  and whose ordinal number is an ordinal  $\rho'$  of  $r'$  less than  $\rho$ . The problem of finding an increasing sequence of ordinals whose upper limit is  $\rho$  and whose ordinal number is an ordinal of  $\tilde{r}$  therefore reduces to the corresponding problem for the smaller ordinal  $\rho'$ . And this reduction continues until we obtain such a sequence or one of the other possible cases arises.

In all other cases we proceed exactly as we did in proving Theorem B<sub>5</sub>.

**COROLLARY 2.** *The sequence  $\tilde{r}$  of those ordinals which occur in the first place in the successive derived sequences of  $r$  is a reduction sequence.*

For  $\tilde{r}$  contains  $\tilde{r}$ .

**COROLLARY 3.** *The first derived sequence  $\tilde{r}'$  of  $\tilde{r}$ , and therefore all the derived sequences of  $\tilde{r}$ , are reduction sequences.*

Since  $\tilde{r}$  is a reduction sequence, we can assign to every ordinal  $\kappa$  of the second kind in the second ordinal class an increasing sequence  $v_\kappa$  of ordinals

whose upper limit is  $\kappa$  and whose ordinal number  $\alpha$  is either  $\omega$  or an ordinal of  $\bar{r}$ . If  $\alpha$  is not an ordinal of  $\bar{r}'$ , it occurs in the  $\kappa'$ 'th place in  $\bar{r}$ , where  $\kappa'$  is an ordinal of the second kind less than  $\kappa$ . The problem of finding an increasing sequence of ordinals whose upper limit is  $\kappa$  and whose ordinal number is either  $\omega$  or an ordinal of  $\bar{r}'$  therefore reduces to the corresponding problem for the smaller ordinal  $\kappa'$ , and this reduction continues until an ordinal  $\kappa^{(k)}$  is obtained which is either  $\omega$  or an ordinal of  $\bar{r}'$ .

**COROLLARY 4.** *The first derived sequence  $\bar{r}'$  of  $\bar{r}$ , and therefore all the derived sequences of  $\bar{r}$ , are reduction sequences.*

For  $\bar{r}$  contains  $\bar{r}'$ , and therefore  $\bar{r}'$  contains  $\bar{r}''$ .

It is possible that it can be proved that every internally closed sequence of ordinal number  $\Omega$  and consisting of ordinals of the second kind is a reduction sequence. At any rate, we are not able to show the contrary.

In particular, the question naturally arises in this connection whether there exists a reduction sequence whose first derived sequence is not a reduction sequence. It is clear, in view of Theorem B<sub>5</sub>, that if such a reduction sequence existed, it could not be the first derived sequence of any sequence. But it does not follow from this alone that such reduction sequences do not exist. In fact it is possible to find internally closed sequences of ordinal number  $\Omega$  in the second ordinal class which are not first derived sequences of other sequences. In constructing an example we have only to choose  $2\omega$  as the first ordinal of the sequence, because, in any order preserving one-to-one correspondence between the set of ordinals less than  $2\omega$  and a subset of them, the ordinal  $\omega$  necessarily corresponds to itself. As an example of a sequence which not only is not the first derived sequence of any sequence but retains that property no matter how many ordinals are omitted from the beginning of it, we may take the sequence  $s$  of the ordinals  $(2\omega)^\alpha$  arranged in order of magnitude, where  $\alpha$  takes on all values less than  $\Omega$  except the value 0. The sequence  $s$  and its first derived sequence (namely the sequence of  $\epsilon$ -numbers) are, however, both reduction sequences.

**10. Consequences of Postulate C.** **DEFINITION.** Under Postulate C there is an ordinal  $\phi$  of the second ordinal class such that there exists no assignment of a unique fundamental sequence to every ordinal of the second kind less than  $\phi$ . The ordinal  $v_1$  is the least such ordinal  $\phi$  in the second ordinal class.

**THEOREM C<sub>1</sub>.** *The ordinal  $v_1$  is an ordinal of the second kind.*

For if  $v_1$  were an ordinal of the first kind, there would be an ordinal  $\alpha$  which next preceded  $v_1$ , and there would exist an assignment of a unique

fundamental sequence to every ordinal of the second kind less than  $\alpha$ . Then, in order to obtain an assignment of a unique fundamental sequence to every ordinal of the second kind less than  $v_1$ , we would have to make at most one arbitrary choice, namely a choice of a fundamental sequence for  $\alpha$  if  $\alpha$  were of the second kind. Since a single arbitrary choice is always permissible, this would lead to a contradiction with the definition of  $v_1$ . Therefore  $v_1$  is an ordinal of the second kind.

The way in which Postulate C involves a denial of the axiom of choice for sets of  $\aleph_0$  classes can be made clear in this connection by proposing the following argument which purports to show that Postulate C leads to a contradiction. Let  $\alpha_0, \alpha_1, \alpha_2, \dots$  be a fundamental sequence for  $v_1$ . Since  $\alpha_0$  is less than  $v_1$  it follows at once from the definition of  $v_1$  that there exists an assignment  $A_0$  of a unique fundamental sequence to every ordinal of the second kind less than  $\alpha_0$ , and similarly that there exists an assignment  $A_1$  of a unique fundamental sequence to every ordinal of the second kind less than  $\alpha_1$ , an assignment  $A_2$  of a unique fundamental sequence to every ordinal of the second kind less than  $\alpha_2$ , and so on. Then, in order to obtain an assignment of a unique fundamental sequence to every ordinal of the second kind less than  $v_1$ , we may use  $A_0$  for ordinals less than  $\alpha_0$ ,  $A_1$  for ordinals less than  $\alpha_1$  and not less than  $\alpha_0$ ,  $A_2$  for ordinals less than  $\alpha_2$  and not less than  $\alpha_1$ , and so on.

This argument fails to obtain a contradiction from Postulate C because the assignments  $A_0, A_1, A_2, \dots$  are each of them only one of the many existing assignments of the required character, so that there is an element of arbitrary choice involved in fixing upon the particular assignment  $A_0$ , another in fixing upon  $A_1$ , and so on. The use of all the assignments,  $A_0, A_1, A_2, \dots$ , simultaneously, therefore, involves an appeal to Zermelo's assumption, and this is, of course, inadmissible in this connection.

**THEOREM C<sub>2</sub>.** *If  $\alpha$  is an ordinal of the second ordinal class less than  $v_1$ , the cardinal number corresponding to  $\alpha$  is  $\aleph_0$ .*

Since  $\alpha$  is less than  $v_1$  it is possible to assign to every ordinal  $\beta$  of the second kind less than or equal to  $\alpha$  a fundamental sequence  $u_\beta$ . The proof of Theorem A<sub>1</sub> can, therefore, be used here without change.

**THEOREM C<sub>3</sub>.** *The cardinal number corresponding to  $v_1$  is not  $\aleph_0$ .*

For suppose that the set of ordinals less than  $v_1$  could be arranged in a sequence  $t$  of ordinal number  $\omega$ . Then for any ordinal  $\kappa$  less than  $v_1$  we could obtain a fundamental sequence by omitting from  $t$ , first all ordinals which were not less than  $\kappa$ , then all ordinals which did not have the property of being

greater than every ordinal less than  $\kappa$  which preceded them in  $t$ . This would enable us to assign a fundamental sequence to every ordinal  $\kappa$  of the second kind less than  $v_1$ , contrary to the definition of  $v_1$ .

COROLLARY. *The cardinal number corresponding to  $v_1$  is  $\aleph_1$ .*

We have pointed out in §2 above that the definition of the second ordinal class which we are using differs from Cantor's definition. Under the latter the second ordinal class consists of all those ordinals to which the corresponding cardinal number is  $\aleph_0$ . In connection with Postulate C, these would be the ordinals less than  $v_1$  and not less than  $\omega$ , whereas, under the definition which we are using, the second ordinal class contains ordinals greater than  $v_1$ .

The convenience of a definition in terms of order, such as the one which we are using, lies in the fact that it enables us to use unchanged many known theorems about the second ordinal class which have been proved by means of order properties and therefore apply to the more extensive class of ordinals rather than to the ordinals between  $\omega$  and  $v_1$ . An example is the theorem that if  $f$  is a continuous increasing function\* defined for the set of ordinals less than  $\Omega$  and its value is always an ordinal less than  $\Omega$  then the first derived function\* of  $f$  exists.† The truth of this theorem is not affected by our choice among the postulates A, B, C, if we adhere to the definition of  $\Omega$  which we have given. But, as we shall prove in Theorem C<sub>3</sub> below, Postulate C implies the falsehood of the theorem just stated if we take  $\Omega$  to be the ordinal which we have called  $v_1$ , as we should have to if we used Cantor's definition of the second ordinal class. Another example is the theorem, to which we shall refer below, that every ordinal of the second ordinal class can be expressed in Cantor's normal form. This theorem is true not only of ordinals of the second ordinal class less than  $v_1$  but of all the ordinals of the second ordinal class as we have defined it, because Cantor's proof is equally applicable to ordinals less than  $v_1$  and to ordinals of the second ordinal class greater than  $v_1$ .

Nevertheless it is true that, in many ways, the ordinal  $v_1$  plays, in connection with Postulate C, a rôle similar to that played by  $\Omega$  in connection with Postulates A and B.

THEOREM C<sub>4</sub>. *There exists a denumerable set of subsets  $S_0, S_1, S_2, \dots$  of the continuum such that there is no choice of one element out of each of the sets  $S_0, S_1, S_2, \dots$ .*

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\*For definitions, see below.

†O. Veblen, loc. cit., p. 283.

By Theorem 1, there exists a one-to-one correspondence between the continuum and the set of well-ordered rearrangements of the positive integers. Let  $K$  be such a one-to-one correspondence. Let  $\alpha_0, \alpha_1, \alpha_2, \dots$  be a fundamental sequence for  $v_1$ . And let  $S_i$  be the set of those numbers of the continuum which correspond under  $K$  to well-ordered rearrangements of the positive integers of ordinal number  $\alpha_i$ . Then there is no choice of one element out of each of the sets  $S_i$ , because, if there were, this would lead to a choice, for each ordinal  $\alpha_i$ , of a well-ordered rearrangement of the positive integers of ordinal number  $\alpha_i$ , and this would lead in turn to a choice, for each ordinal  $\alpha_i$ , of an arrangement of the ordinals less than  $\alpha_i$  in a sequence of ordinal number  $\omega$ , and then, by the method of Theorem A<sub>1</sub>, we could obtain an arrangement of the ordinals less than  $v_1$  in a sequence of ordinal number  $\omega$ , contrary to Theorem C<sub>3</sub>.

**COROLLARY.** *The continuum cannot be well-ordered.*

**THEOREM C<sub>6</sub>.** *There exists a set  $T$  of points of the continuum and a set  $I$  of intervals which covers  $T$  such that no denumerable subset of  $I$  covers\*  $T$ .*

On account of the possibility of setting up a one-to-one correspondence between the continuum and the segment  $(0, 1/2)$  of the continuum, the sets  $S_i$  of the preceding theorem can be so chosen that all their elements lie between 0 and  $1/2$ . Let the sets  $S_i$  be chosen in this way, and let  $T$  consist of the points  $0, 1, 2, 3, \dots$ . Let  $I_i$  consist of all intervals  $(i-s_i, i+s_i)$ , where  $s_i$  is an element of  $S_i$ . Then there is a one-to-one correspondence between  $I_i$  and  $S_i$ . Let  $I$  be the sum of all the sets  $I_i$ , so that  $I$  contains every interval which occurs in any one of the sets  $I_i$ .

It is clear that  $I$  covers  $T$ . Suppose that some denumerable subset  $J$  of  $I$  covered  $T$ . Then  $J$  would contain at least one interval in common with

\*The contrary of this theorem has been proved by W. H. Young with the aid of the axiom of choice. See Proceedings of the London Mathematical Society, vol. 35 (1902), pp. 384-388, and also W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, pp. 38-40.

See also E. Lindelöf, *Sur quelques points de la théorie des ensembles*, Comptes Rendus, vol. 137 (1903), pp. 697-700, and *Remarques sur un théorème fondamental de la théorie des ensembles*, Acta Mathematica, vol. 29 (1905), pp. 187-189. Lindelöf's theorem is stronger than that given by Young in that it applies to space of any finite number of dimensions and weaker in that there must be a one-to-one correspondence between the points of  $I$  and the intervals (or  $n$ -spheres) of  $T$ , each point being at the center of the corresponding interval (or  $n$ -sphere). The axiom of choice is necessary to the proof.

The falsity of Lindelöf's theorem for one dimension can be proved as a consequence of Postulate C by means of the following modification of the proof of Theorem C<sub>4</sub>. Let  $T$  consist of all points  $i+(1/2)s_i$ ,  $i=0, 1, 2, 3, \dots$ , and to the point  $i+(1/2)s_i$  let correspond the interval  $(i, i+s_i)$ ,  $I_i$  consisting of all intervals  $(i, i+s_i)$  for a fixed value of  $i$ , and  $I$  being the sum of the sets  $I_i$ .

each of the sets  $I_i$ . And  $J$  could be arranged in a sequence of ordinal number  $\omega$ , and for each set  $I_i$  could be chosen the first interval of  $J$  which belonged also to  $I_i$ . And this would effect a choice of one interval out of each of the sets  $I_i$ , corresponding to which there would be a choice of one element out of each of the sets  $S_i$ , contrary to the preceding theorem.

COROLLARY. *No subset of  $I$  which can be well-ordered covers  $T$ .*

DEFINITION. An internally closed sequence  $r$  of ordinals of the second kind less than  $v_1$  all distinct and arranged in their natural order is a *reduction sequence* if there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $v_1$  of a sequence  $v_\kappa$  of distinct ordinals arranged in their natural order, such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number  $v_\kappa$  is either  $\omega$  or one of the ordinals of  $r$ .

We shall use the same definition of a derived sequence as that given in the preceding section.

DEFINITION. A function  $f$  defined for all ordinals less than a certain ordinal  $\Xi$ , the value of  $f$  being always an ordinal, is a *continuous increasing function*,\* if, for every pair of ordinals  $\xi_1$  and  $\xi_2$ , both less than  $\Xi$ , such that  $\xi_1$  is less than  $\xi_2$ , it is true that  $f(\xi_1)$  is less than  $f(\xi_2)$ , and the set  $t$  of all ordinals  $f(\xi)$ , where  $\xi$  takes on all values less than  $\Xi$ , forms an internally closed sequence when the ordinals are arranged in their natural order. The  $\alpha$ th *derived function*\* of  $f$  is the continuous increasing function of  $\xi$ ,  $f(\xi, \alpha)$ , determined by the one-to-one correspondence between the  $\alpha$ th derived sequence  $t^\alpha$  of  $t$  and the whole or a segment of the sequence of all ordinals less than  $\Xi$ .

THEOREM C<sub>6</sub>. *The ordinal  $v_1$  is an  $\epsilon$ -number and occupies the  $v_1$ th place in the sequence of  $\epsilon$ -numbers arranged in their natural order.*

Since  $\omega^\xi$  is a continuous increasing function of  $\xi$ , we know that  $\omega^{v_1}$  is greater than or equal to  $v_1$ .

Suppose that  $\omega^{v_1}$  is greater than  $v_1$ . Then, since the sequence of ordinals  $\omega^\xi$  in their natural order is internally closed, there is a greatest ordinal  $\alpha$  such that  $\omega^\alpha$  is not greater than  $v_1$ . And  $\alpha$  is less than  $v_1$ . Assign to every ordinal  $\beta$  of the second kind less than or equal to  $\alpha$  a fundamental sequence  $\beta_0, \beta_1, \beta_2, \dots$ . Since the ordinals  $\omega^\xi$  are the self-residual ordinals, every ordinal  $\kappa$  of the second kind less than  $v_1$  can be written in a unique way in the form  $\gamma + \omega^\beta$ , where  $\gamma$  has its least possible value, which may be 0, and  $\beta$  is not greater than  $\alpha$ . If  $\beta$  is of the first kind it has an immediate predecessor

\* O. Veblen, loc. cit.

$\zeta$ , and a fundamental sequence for  $\kappa$  is  $\gamma + \omega^\zeta, \gamma + 2\omega^\zeta, \gamma + 3\omega^\zeta, \dots$  (if  $\zeta$  is equal to 0,  $\omega^\zeta$  is to be taken equal to 1). If  $\beta$  is of the second kind, the sequence  $\gamma + \omega^{\beta_0}, \gamma + \omega^{\beta_1}, \gamma + \omega^{\beta_2}, \dots$  is a fundamental sequence for  $\kappa$ . In this way we are able to assign a fundamental sequence to every ordinal  $\kappa$  of the second kind less than  $v_1$ , contrary to the definition of  $v_1$ .

Therefore  $\omega^\alpha$  is equal to  $v_1$ . Therefore  $v_1$  is an  $\epsilon$ -number.

In the sequence of  $\epsilon$ -numbers in their natural order let the  $\alpha$ th place be the place occupied by  $v_1$ . Then  $\alpha$  is less than or equal to  $v_1$ .

Suppose that  $\alpha$  is less than  $v_1$ . To every ordinal  $\theta$  of the second kind less than  $\alpha$  assign a fundamental sequence  $\theta_0, \theta_1, \theta_2, \dots$ . Every ordinal  $\kappa$  of the second kind less than  $v_1$  can be written in a unique way in the form  $\gamma + \omega^{\kappa'}$ , where  $\gamma$  has its least possible value, which may be 0, and  $\kappa'$  is not greater than  $\kappa$ .

If  $\kappa'$  is an  $\epsilon$ -number, so that  $\kappa' = \omega^{\epsilon_\theta} = \epsilon_\theta$ , then  $\theta$  is less than  $\alpha$ . If  $\theta$  is of the second kind, a fundamental sequence for  $\kappa$  is  $\gamma + \epsilon_{\theta_0}, \gamma + \epsilon_{\theta_1}, \gamma + \epsilon_{\theta_2}, \dots$ . If  $\theta$  is of the first kind it has an immediate predecessor  $\delta$ , and a fundamental sequence for  $\kappa$  is\*

$$\gamma + \epsilon_\delta + 1, \gamma + \omega^{\epsilon_\delta + 1}, \gamma + \omega^{\omega^{\epsilon_\delta + 1}}, \dots$$

If  $\kappa'$  is of the first kind it has an immediate predecessor  $\zeta$ , and a fundamental sequence for  $\kappa$  is  $\gamma + \omega^\zeta, \gamma + 2\omega^\zeta, \gamma + 3\omega^\zeta, \dots$ .

If  $\kappa'$  is of the second kind but is not an  $\epsilon$ -number, then the problem of finding a fundamental sequence for  $\kappa$  reduces to that of finding a fundamental sequence for the ordinal  $\kappa'$ , less than  $\kappa$ , and this reduction continues in the same way until an ordinal  $\kappa^{(k)}$  is obtained which either is of the first kind or is an  $\epsilon$ -number.

In this way we are able to assign a fundamental sequence to every ordinal  $\kappa$  of the second kind less than  $v_1$ , contrary to the definition of  $v_1$ .

Therefore  $\alpha$  is equal to  $v_1$ .

**COROLLARY 1.** *The ordinal  $v_1$  is a self-residual ordinal.*

**COROLLARY 2.** *The sequence of self-residual ordinals less than  $v_1$  arranged in their natural order, and its first derived sequence, the sequence of  $\epsilon$ -numbers less than  $v_1$  arranged in their natural order, are reduction sequences.*

**THEOREM C<sub>7</sub>.** *If the first derived sequence of a reduction sequence  $r$  is a reduction sequence, then the ordinal number of  $r$  is  $v_1$ , and the first  $v_1$  derived sequences of  $r$  exist and are reduction sequences of ordinal number  $v_1$ .*

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\*G. Cantor, loc. cit., zweiter Artikel, p. 243.

If  $\nu$  is an ordinal less than  $v_1$  we can prove by the same argument as that used in proving Theorem B<sub>5</sub> that  $r^\nu$  is a reduction sequence, and this argument applies even if we suppose  $r^\nu$  empty. And then, as soon as we have proved that  $r^\nu$  is a reduction sequence, it follows from Postulate C that  $r^\nu$  is not empty. Therefore the first  $v_1$  derived sequences of  $r$  exist and are reduction sequences.

It remains to prove that  $r$  and its first  $v_1$  derived sequences are all of ordinal number  $v_1$ .

The ordinal number  $\alpha$  of  $r^\nu$  cannot be greater than  $v_1$  because  $r^\nu$  consists entirely of ordinals less than  $v_1$ . Suppose that  $\alpha$  is less than  $v_1$ . Then  $r^{\nu+1}$  contains no ordinal greater than or equal to  $\alpha$ . Therefore we can assign a fundamental sequence  $\rho_0, \rho_1, \rho_2, \dots$  to every ordinal  $\rho$  in  $r^{\nu+1}$ . But to every ordinal  $\kappa$  of the second kind less than  $v_1$  we can assign an increasing sequence  $\kappa_0, \kappa_1, \kappa_2, \dots, \kappa_\omega, \dots$  of ordinals whose upper limit is  $\kappa$  and whose ordinal number is either  $\omega$  or an ordinal  $\rho$  of  $r^{\nu+1}$ . If the ordinal number of this increasing sequence is not  $\omega$ , a fundamental sequence for  $\kappa$  is  $\kappa_{\rho_0}, \kappa_{\rho_1}, \kappa_{\rho_2}, \dots$ . Therefore we can assign a fundamental sequence to every ordinal  $\kappa$  of the second kind less than  $v_1$ , contrary to the definition of  $v_1$ .

Therefore the ordinal number  $\alpha$  of  $r^\nu$  is  $v_1$ . And in the same way we can prove that the ordinal number of  $r$  is  $v_1$ .

COROLLARY 1. Let  $f(\xi) = \omega^\xi$ . Then, if  $\alpha$  is less than  $v_1$ ,  $f(v_1, \alpha) = v_1$ .

COROLLARY 2. If  $f(\xi) = \omega^\xi$ , then  $f(0, v_1) = v_1$ .

COROLLARY 3. The sequence  $\bar{r}$  of those ordinals which occur in the first place in the successive derived sequences of  $r$  is a reduction sequence of ordinal number  $v_1$ , and the first  $v_1$  derived sequences of  $\bar{r}$  are reduction sequences of ordinal number  $v_1$ .

The proof of this is the same as that of the corollaries to Theorem B<sub>5</sub>.

COROLLARY 4. Let  $f(\xi) = \omega^\xi$  and  $\phi(\xi) = f(0, \xi)$ . Then if  $\alpha$  is less than  $v_1$ ,  $\phi(v_1, \alpha) = v_1$ , and  $\phi(0, v_1) = v_1$ .

THEOREM C<sub>8</sub>. There exists a continuous increasing function defined for the set of ordinals less than  $v_1$ , the value of which is always an ordinal less than  $v_1$ , and the first derived function of which does not exist.

Since  $v_1$  is of the second kind, there exists a fundamental sequence  $\alpha_0, \alpha_1, \alpha_2, \dots$  for  $v_1$ . Let  $f(\xi) = \omega^\xi$ , and let  $\beta_i = f(0, \alpha_i + 1)$  for  $i = 0, 1, 2, \dots$ . Then the upper limit of the sequence  $\beta_0, \beta_1, \beta_2, \dots$  is  $v_1$ . And  $\beta_i$  is the least value of  $\xi$  such that  $f(\xi, \alpha_i) = \xi$ .

Let a function  $F$  be defined as follows. If  $0 \leq \xi \leq \beta_0$ ,  $F(\xi) = f(\xi, \alpha_1)$ , and if  $\beta_i < \xi \leq \beta_{i+1}$ ,  $F(\xi) = f(\xi, \alpha_{i+2})$ . Then  $F$  is a continuous increasing function defined for the set of ordinals less than  $v_1$ , and its value is always an ordinal less than  $v_1$ , but there is no ordinal  $\xi$  less than  $v_1$  such that  $F(\xi) = \xi$ , and therefore the first derived function of  $F$  does not exist.

**11. Properties of  $v$ -numbers.** DEFINITION. The  $v$ -numbers are those ordinals  $\kappa$  of the second ordinal class, greater than  $\omega$ , which have the property that the cardinal number corresponding to  $\kappa$  is greater than that corresponding to any ordinal less than  $\kappa$ .

If Postulate C is denied, it follows that  $v$ -numbers do not exist (Theorems  $A_1$  and  $B_1$ ).

If Postulate C is accepted, we are assured of the existence of at least one  $v$ -number, namely  $v_1$ . Postponing the question how many  $v$ -numbers the second ordinal class contains, we are able to say that, in any case, the  $v$ -numbers arranged in order of magnitude form a well-ordered sequence, so that the  $\alpha$ th ordinal of this sequence, counting from  $v_1$  as the first ordinal of the sequence, may be indicated by the symbol  $v_\alpha$ .

The following theorems are consequences of Postulates 1-5 alone, but since they become vacuous if Postulate A or Postulate B is accepted, we think of them as belonging with Postulate C.

**THEOREM  $C_9$ .** *The  $v$ -numbers are ordinals of the second kind.*

For let  $\alpha+1$  be any ordinal of the first kind in the second ordinal class. Then the set of ordinals of the second ordinal class less than  $\alpha+1$  can be arranged in a sequence of ordinal number  $\alpha$  by placing  $\alpha$  first and letting the remaining ordinals follow in their natural order, thus,  $\alpha, 0, 1, 2, 3, \dots, \omega, \omega+1, \dots$ . But if  $\alpha+1$  were a  $v$ -number, the set of ordinals of the second ordinal class less than  $\alpha+1$  could not be arranged in a sequence of ordinal number less than  $\alpha+1$ . Therefore  $\alpha+1$  is not a  $v$ -number.

**THEOREM  $C_{10}$ .** *Given any increasing sequence  $s$  of  $v$ -numbers,  $v_{\alpha_0}, v_{\alpha_1}, v_{\alpha_2}, \dots$ , of ordinal number  $\omega$ , the upper limit  $v_\alpha$  of  $s$  is a  $v$ -number.*

The cardinal number corresponding to  $v_\alpha$  cannot be equal to that corresponding to any ordinal  $v_{\alpha_i}$  of  $s$ , because, if it were, it would be less than that corresponding to  $v_{\alpha_{i+1}}$ . Therefore the cardinal number corresponding to  $v_\alpha$  is greater than that corresponding to any ordinal of  $s$ . Therefore the cardinal number corresponding to  $v_\alpha$  is greater than that corresponding to any less ordinal. Therefore  $v_\alpha$  is a  $v$ -number.

**THEOREM  $C_{11}$ .** *The  $v$ -numbers are  $\epsilon$ -numbers.*

We have already shown that  $v_1$  is an  $\epsilon$ -number. We shall show by transfinite induction that the remaining  $v$ -numbers (if any other  $v$ -numbers exist) are also  $\epsilon$ -numbers.

If  $v_\alpha$  is a  $v$ -number which is also an  $\epsilon$ -number, then the next following  $v$ -number  $v_{\alpha+1}$  (if it exist) is also an  $\epsilon$ -number. For suppose  $v_{\alpha+1}$  is not an  $\epsilon$ -number. Then it can be written in Cantor's normal form\*:

$$a_0\omega^{v_0} + a_1\omega^{v_1} + \dots + a_n\omega^{v_n}$$

where  $v_{\alpha+1} > v_0 > v_1 > \dots > v_n$ , the coefficients  $a_i$  are finite ordinals, and the sum contains a finite number of terms in all. Since  $v_\alpha$  is an  $\epsilon$ -number,  $\omega^{v_\alpha} = v_\alpha$ . Therefore  $v_0$  is greater than  $v_\alpha$ . Therefore  $v_0 + 1$  is greater than  $v_\alpha$ . And, since  $v_0$  is less than  $v_{\alpha+1}$ ,  $v_0 + 1$  is also less than  $v_{\alpha+1}$ , by Theorem C<sub>0</sub>. Consequently, since  $v_{\alpha+1}$  is the next  $v$ -number after  $v_\alpha$ , the set of ordinals less than  $v_0 + 1$  can be put into one-to-one correspondence with the set of ordinals less than  $v_\alpha$ . Let such a one-to-one correspondence be set up, and if  $\kappa$  is any ordinal of the set of ordinals less than  $v_0 + 1$ , let  $\kappa'$  be the corresponding ordinal of the set of ordinals less than  $v_\alpha$ . Now every ordinal less than  $v_{\alpha+1}$  can be written in Cantor's normal form,  $\sum_i b_i \omega^{\mu_i}$ , where  $v_0 + 1 > \mu_0 > \mu_1 > \dots$ , the coefficients  $b_i$  are finite ordinals, and the sum contains a finite number of terms in all. The understanding is that one of the exponents  $\mu_i$  may have the value 0, and that  $\omega^0$  is to be taken equal to 1. To the ordinal  $\sum_i b_i \omega^{\mu_i}$ , less than  $v_{\alpha+1}$ , let correspond the ordinal  $\sum_i b_i \omega^{\mu'_i}$ , where the terms of the sum are to be arranged in order of magnitude, the greatest first. Then  $\sum_i b_i \omega^{\mu'_i}$  is less than  $v_\alpha$ , because  $v_\alpha = \omega^{v_\alpha}$ , and all the exponents  $\mu'_i$  are less than  $v_\alpha$ . We have, accordingly, set up in this way a one-to-one correspondence between the set of all ordinals less than  $v_{\alpha+1}$  and a certain set of ordinals less than  $v_\alpha$ . But this is impossible, because the cardinal number corresponding to  $v_{\alpha+1}$  is greater than that corresponding to  $v_\alpha$ . Therefore the supposition that  $v_{\alpha+1}$  was not an  $\epsilon$ -number was incorrect.

If every ordinal of the increasing sequence of  $v$ -numbers  $v_{\beta_0}, v_{\beta_1}, v_{\beta_2}, \dots$  is an  $\epsilon$ -number, the upper limit  $v_\beta$  of the sequence is an  $\epsilon$ -number, because the  $\epsilon$ -numbers form in order of magnitude an internally closed sequence.

Therefore, by transfinite induction, every  $v$ -number is an  $\epsilon$ -number.

**COROLLARY.** *If  $\alpha$  is any ordinal of the second ordinal class, the cardinal number corresponding to  $\alpha^\omega$  is the same as that corresponding to  $\alpha$ . And, therefore, if  $\beta$  is any ordinal greater than  $\alpha$  and less than  $\alpha^\omega$ , the cardinal number corresponding to  $\beta$  is the same as that corresponding to  $\alpha$ .*

\*G. Cantor, loc. cit., zweiter Artikel, p. 237.

For the least  $\epsilon$ -number  $\epsilon_\beta$  greater than  $\alpha$  is the upper limit of the sequence\*  $\alpha+1, \omega^{\alpha+1}, \omega^{\omega^{\alpha+1}}, \dots$  and is, therefore, greater than  $\omega^{\omega^{\alpha+1}}$ , or  $(\omega^{\omega^\alpha})^\omega$ . And  $\omega^{\omega^\alpha}$  is greater than or equal to  $\alpha$ . Therefore  $\epsilon_\beta$  is greater than  $\alpha^\omega$ . But the least  $v$ -number greater than  $\alpha$  is greater than or equal to  $\epsilon_\beta$  and therefore greater than  $\alpha^\omega$ . Therefore the cardinal number corresponding to  $\alpha^\omega$  is the same as that corresponding to  $\alpha$ .

12. **Postulates F and G.** In connection with Postulate C there appear two possibilities, which we shall state as Postulates F and G, inconsistent with each other, but each apparently consistent with Postulates 1-5 and C. These possibilities are the following :

**F.** *If  $\psi$  is any ordinal of the second ordinal class, there is some ordinal  $\alpha$  of the second ordinal class, such that there exists no assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $\psi$ .*

**G.** *There is an ordinal  $\psi$  of the second ordinal class such that, given any ordinal  $\alpha$  of the second ordinal class, there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $\psi$ .*

Postulate F is stated in such a way that it implies Postulate C, but Postulate G does not.

We shall examine briefly the consequences of each of the postulates just stated when taken in conjunction with Postulates 1-5 and C, taking the same experimental attitude as that which we took in the case of Postulates A, B, and C.

13. **Consequences of Postulate F.** **THEOREM F<sub>1</sub>.** *If  $v_\eta$  is any  $v$ -number, there exists a  $v$ -number greater than  $v_\eta$ .*

For suppose the contrary. Then there exists a greatest  $v$ -number,  $v_\beta$ . Let  $\psi$  be the ordinal  $v_\beta+1$  and let  $\alpha$  be any ordinal of the second ordinal class. Then the set of all ordinals less than  $\alpha$  can be arranged in a sequence  $t_\alpha$  of ordinal number less than  $\psi$ . Then there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $\psi$ . For we could choose  $v_\kappa$  to be the sequence obtained by omitting from  $t_\alpha$ , first all ordinals not less than  $\kappa$ , and then all ordinals which do not have the property of being greater than every ordinal less than  $\kappa$  which precedes them in  $t_\alpha$ .

This, however, is contrary to Postulate F. Therefore if  $v_\eta$  is any  $v$ -number, there exists a  $v$ -number greater than  $v_\eta$ .

\*G. Cantor, loc. cit., zweiter Artikel, p. 243.

THEOREM F<sub>2</sub>. *The sequence of the  $v$ -numbers of the second ordinal class arranged in order of magnitude is an internally closed sequence of ordinal number  $\Omega$ .*

This follows at once from the preceding theorem and Theorem C<sub>10</sub>.

COROLLARY. *The cardinal number corresponding to  $\Omega$  is  $\aleph_0$ .*

14. **Consequences of Postulate G.** Turning now to the consequences of Postulates C and G taken together, we recall that, in accordance with Postulate G, there exist ordinals  $\psi$  in the second ordinal class such that, given any ordinal  $\alpha$  of the second ordinal class, there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $\psi$ . Let  $T$  be the least such ordinal  $\psi$ . Then

THEOREM CG<sub>1</sub>. *The ordinal  $T$  is an ordinal of the second kind.*

For suppose that  $T$  is an ordinal of the first kind. Then there exists an ordinal  $\beta$  such that  $T$  is equal to  $\beta + 1$ .

In accordance with the definition of  $T$ , given any ordinal  $\alpha$  of the second ordinal class, there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $T$  and therefore less than or equal to  $\beta$ .

If  $\beta$  is an ordinal of the first kind the ordinal number  $v_\kappa$  cannot be equal to  $\beta$ , because only those sequences in which there is no greatest ordinal have an upper limit. Therefore in this case the ordinal number of  $v_\kappa$  is always less than  $\beta$ , contrary to the definition of  $T$ .

If  $\beta$  is an ordinal of the second kind it has a fundamental sequence  $\beta_0, \beta_1, \beta_2, \dots$ . Those sequences  $v_\kappa$  which are of ordinal number  $\beta$  can then be replaced by sequences  $v'_\kappa$  of ordinal number  $\omega$  obtained by omitting from  $v_\kappa$  all ordinals except those in the positions  $\beta_0, \beta_1, \beta_2, \dots$ . And in this way we obtain again a contradiction of the definition of  $T$ .

Therefore  $T$  is an ordinal of the second kind.

THEOREM CG<sub>2</sub>. *The ordinal  $T$  is a  $v$ -number.*

Suppose that to some ordinal  $\beta$  less than  $T$  corresponds the same cardinal number as to  $T$ . Then the set of ordinals less than  $T$  can be rearranged in a sequence of ordinal number  $\beta$ . Choosing a particular such rearrangement  $t$  of the set of ordinals less than  $T$ , we have a uniform method of rearranging any given well-ordered sequence  $s$  of ordinal number greater than  $\beta$  but not greater than  $T$  in a well-ordered sequence of ordinal number less than or equal

to  $\beta$ , because there is a one-to-one correspondence between  $s$  and the whole or a segment of the sequence  $u$  of ordinals less than  $T$  arranged in order of magnitude, so that the rearrangement  $l$  of  $u$  determines the desired rearrangement of  $s$ .

Now in accordance with the definition of  $T$ , given any ordinal  $\alpha$  of the second ordinal class, there exists an assignment to every ordinal  $\kappa$  of the second kind less than  $\alpha$  of an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $T$ . In accordance with the preceding paragraph we can rearrange  $v_\kappa$  as a well-ordered sequence of ordinal number less than or equal to  $\beta$ , and by omitting from the rearranged sequence all ordinals which do not have the property of being greater than every ordinal which precedes them in the sequence we obtain an increasing sequence  $w_\kappa$  of ordinals such that the upper limit of  $w_\kappa$  is  $\kappa$  and the ordinal number of  $w_\kappa$  is less than or equal to  $\beta$  and therefore less than  $\beta+1$ .

Since, by hypothesis,  $\beta$  is less than  $T$  so that  $T$  cannot be less than  $\beta+1$ , it follows, in view of the definition of  $T$ , that  $T$  is equal to  $\beta+1$ . This, however, is contrary to Theorem  $CG_1$ .

Therefore there is no ordinal  $\beta$  less than  $T$  such that the same cardinal number corresponds to  $\beta$  as to  $T$ .

But it follows at once from Postulate C that  $T$  is greater than  $\omega$ . Therefore  $T$  is a  $v$ -number.

**THEOREM  $CG_3$ .** *The ordinal  $T$  is the greatest  $v$ -number.*

Let  $\alpha$  be an ordinal of the second ordinal class, greater than  $T$ . Assign to every ordinal  $\kappa$  of the second kind which is less than or equal to  $\alpha$  an increasing sequence  $v_\kappa$  of ordinals such that the upper limit of  $v_\kappa$  is  $\kappa$  and the ordinal number of  $v_\kappa$  is less than  $T$ .

The set of ordinals which precede  $T$  form, when arranged in their natural order, a sequence of ordinal number  $T$ . With this as a starting point assign to the ordinals which follow  $T$ , one by one in order, an arrangement of all preceding ordinals in a sequence of ordinal number  $T$ , in the following way.

When we have assigned an arrangement in a sequence  $l$  of ordinal number  $T$  of all ordinals which are less than an ordinal  $\gamma$ , an arrangement in a sequence of ordinal number  $T$  of all ordinals which are less than  $\gamma+1$  is obtained by placing  $\gamma$  before  $l$ .

When we have assigned to every ordinal  $\zeta$  which is less than an ordinal  $\beta$  of the second kind an arrangement in a sequence  $l_\zeta$  of ordinal number  $T$  of all ordinals which are less than  $\zeta$ , the sequences  $l_{\beta_0}, l_{\beta_1}, l_{\beta_2}, \dots, l_{\beta_\omega}, \dots$ , where  $\beta_0, \beta_1, \beta_2, \dots, \beta_\omega, \dots$  is the sequence  $v_\beta$ , may be written one after the other so as to obtain a sequence  $u_\beta$  of ordinal number not greater than

$T^2$ . In accordance with the corollary of Theorem  $C_{11}$  the set of ordinals less than  $T^2$  can be rearranged in a sequence of ordinal number  $T$ . Choosing a particular such rearrangement  $s$  of the set of ordinals less than  $T^2$ , we have a uniform method of rearranging any sequence  $u_\beta$  in a well-ordered sequence  $w_\beta$  of ordinal number  $T$ , because there is a one-to-one correspondence between  $u_\beta$  and the whole or a segment of the sequence  $v$  of ordinals less than  $T^2$  arranged in their natural order, so that the rearrangement  $s$  of  $v$  determines the desired rearrangement  $w_\beta$  of  $u_\beta$  (the ordinal number of  $w_\beta$  cannot be less than  $T$  on account of the fact that  $T$  is a  $v$ -number). By omitting from  $w_\beta$  all occurrences of any ordinal after the first occurrence, so that a sequence without repetition results, we obtain an arrangement in a sequence of ordinal number  $T$  of all ordinals less than  $\beta$ .

We may prove by induction that this process continues until we obtain an arrangement in a sequence of ordinal number  $T$  of all ordinals less than  $\alpha$ . Therefore the cardinal number corresponding to  $\alpha$  is the same as that corresponding to  $T$ . But  $\alpha$  was any ordinal of the second ordinal class greater than  $T$ . Therefore  $T$  is the greatest  $v$ -number.

**COROLLARY.** *The sequence of the  $v$ -numbers of the second ordinal class arranged in order of magnitude is an internally closed sequence whose ordinal number is an ordinal of the first kind less than  $\Omega$ .*

It should be noted that there is nothing in the preceding to preclude the possibility that  $v_1$  and  $T$  are the same ordinal, in which case  $v_1$  would be the only  $v$ -number.

PRINCETON UNIVERSITY,  
PRINCETON, N. J.

# ON A GENERAL THEOREM CONCERNING THE DISTRIBUTION OF THE RESIDUES AND NON-RESIDUES OF POWERS\*

BY  
J. M. VINOGRADOV

In the present paper I offer a new method for solving some questions regarding the distribution of residues and non-residues of powers.

The difference between the present method and the methods developed in my papers of 1916-18 lies in its entirely elementary character.

The chief idea of this method consists of two different ways of calculating the number of numbers of the form  $\alpha(ax+b)$ , where  $\alpha$  ranges over all the different least positive residues of numbers congruent to  $Ax^n \pmod{p}$  and where  $x$  independently of  $\alpha$  assumes all values  $0, 1, \dots, h-1 (h < p)$ .

I shall deal here with the demonstration of the chief formula only, which gives for the prime  $p$  the number of numbers congruent to  $Ax^n \pmod{p}$  in the progression  $ax+b$ ;  $x=0, 1, \dots, h-1$ , with an approximation of order  $< \sqrt{p} \log p$ .

Other results, such as the law of distribution of the primitive roots, the upper bound  $p^{1/2k}(\log p)^2$ ,  $k=e^{(n-1)/n}$ , for the least positive non-residues of degree  $n$ , modulo  $p$  ( $p-1=nd$ ), and others, follow from this theorem in the same way as in my previous researches on these questions.

In the near future I hope to publish further applications of this method to the demonstration of the chief theorem and to some other important questions of the asymptotic theory of numbers.

LEMMA I. *If  $p$  be a prime number  $>2$ ,  $\alpha$  an integer prime to  $p$ , and  $k$  a positive integer\*, then there exist relatively prime integers  $x$  and  $y$  which satisfy the conditions*

$$\alpha x \equiv y \pmod{p}; \quad 0 < x \leq k; \quad 0 < |y| < p/k.$$

(It is actually proved that  $0 < |y| \leq \frac{p}{k+1}$ .)

Proof. Let us consider the system of congruences

$$\alpha r \equiv \beta_r \pmod{p} \quad (r = 1, 2, \dots, k),$$

the right hand members of which are least positive residues of the left hand ones. Arranging these congruences in such a way that the  $\beta_r$  are ascending,

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\* We must also assume  $k < p$ . The lemma holds also for  $p=2$ .

and adjoining to them the obvious congruence  $\alpha \cdot 0 \equiv p \pmod{p}$ , we obtain the following system:

$$\begin{aligned}\alpha\gamma_1 &\equiv \lambda_1 & (\text{mod } p), \\ \alpha\gamma_2 &\equiv \lambda_2 & (\text{mod } p), \\ &\dots\dots\dots \\ \alpha\gamma_k &\equiv \lambda_k & (\text{mod } p), \\ \alpha \cdot 0 &\equiv p. & (\text{mod } p).\end{aligned}$$

Subtracting one congruence from the other as shown we come to the following system:

$$\begin{aligned}\alpha\gamma_1 &\equiv \lambda_1 & (\text{mod } p), \\ \alpha(\gamma_2 - \gamma_1) &\equiv \lambda_2 - \lambda_1 & (\text{mod } p), \\ \alpha(\gamma_3 - \gamma_2) &\equiv \lambda_3 - \lambda_2 & (\text{mod } p), \\ &\dots\dots\dots \\ \alpha(-\gamma_k) &\equiv p - \lambda_k & (\text{mod } p).\end{aligned}$$

Among the numbers  $\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \dots, p - \lambda_k$  there is certain to be at least one  $\leq p(k+1)^{-1}$ , for the number of these numbers is equal to  $k+1$ , every one of them is greater than 0, and their sum is  $p$ . Among the congruences of the last system there must therefore be at least one of the form

$$\alpha x_1 \equiv y_1 \pmod{p}; \quad 0 < x_1 \leq k; \quad 0 < |y_1| \leq p(k+1)^{-1}.$$

Hence, observing that the numbers  $x_1$  and  $y_1$  can always be reduced to be relatively prime by dividing by their common divisor, we arrive at the conclusion that the lemma is true.

LEMMA\* II. Let  $k$  be any number  $\geq 1$ ,  $q$  a positive integer  $\leq k$ ,  $c$  an integer,  $m$  a positive integer  $\leq kq^{-1}$ ,  $B$  an arbitrary number and  $A$  a number of the form

$$A = \frac{t}{q} + \frac{\theta}{kq}$$

where  $t$  is an integer prime to  $q$ , and  $|\theta| < 1$ . Then, denoting in general by the symbol  $\{z\}$  the fractional part of  $z$  we have

$$S = \sum_{x=0}^{c+mq-1} \{Ax + B\} = \frac{1}{2}mq + \frac{1}{2}\rho(m+1); \quad |\rho| < 1.$$

\* The same lemma somewhat differently formulated is proved in my paper *A new method for obtaining asymptotical expressions of arithmetical functions*, Bulletin of the Russian Academy of Sciences, 1917.

Proof. (i) Let us assume that  $q > 1$ . We have then

$$Ax + B = \frac{tx}{q} + \frac{\theta x}{kq} + B = \frac{tx + f(x)}{q}; f(x) = Bq + \frac{\theta x}{k}.$$

The set of values of the function  $f(x)$  for  $x = c, c+1, \dots, c+mq-1$  forms an arithmetical progression. We shall consider only the case  $\theta \geq 0$ . The case  $\theta < 0$  can be investigated in a similar way. Let  $n = [f(c)]$ . Two cases are possible:

- ( $\alpha$ ) All values of the function  $f(x)$  are less than  $n+1$ .
- ( $\beta$ ) One of them at least  $\geq n+1$ .
- ( $\alpha$ ) Expanding the sum  $S$  in the form of a series of sums

$$(1) \quad S = \sum_{x=c}^{c+q-1} + \sum_{x=c+q}^{c+2q-1} + \dots + \sum_{x=c+mq-q}^{c+mq-1},$$

let us consider one of these sums

$$I_s = \sum_{x=c+sq}^{c+sq+q-1} \left\{ \frac{tx+n+\lambda(x)}{q} \right\},$$

where  $\lambda(x) = f(x) - n$ . Replacing the numbers  $tx+n$  by their least positive residues  $r$ , modulo  $q$  (which is permissible since  $\{z\}$  does not alter by adding to  $z$  an integer), and putting  $\lambda(x) = \nu(r)$  we get

non-negative

$$I_s = \sum_{r=0}^{q-1} \left\{ \frac{r + \nu(r)}{q} \right\} = \sum_{r=0}^{q-1} \frac{r + \nu(r)}{q}.$$

Therefore

$$I_s = \frac{1}{2}q - \frac{1}{2} + \frac{1}{q} \sum_{r=0}^{q-1} \nu(r) = \frac{1}{2}q - \frac{1}{2} + \theta'; 0 \leq \theta' < 1.$$

Hence

$$(2) \quad I_s = \frac{1}{2}q + \frac{1}{2}\rho_s; |\rho_s| \leq 1,$$

$$S = \frac{1}{2}mq + \frac{1}{2}m\rho; |\rho| \leq 1,$$

which proves the case ( $\alpha$ ) of our lemma.

Let us now consider the case ( $\beta$ ). Let  $\sigma$  be the greatest integer that satisfies the condition  $f(c+\sigma q) < n+1$ ; then, putting in the sums  $I_s$  of the series (1) where  $s \leq \sigma$ ,  $\lambda(x) = f(x) - n$ , and in those where  $s > \sigma$ ,  $\lambda(x) = f(x) - n - 1$ , and considering any sum  $I_s$ ,  $s \geq \sigma$ , we shall get  $0 \leq \lambda(x) < 1$  and therefore,

where  $s \leq \sigma$ , we shall have as before the equation (2). Equation (2) holds good also when  $s = \sigma$ , if  $\lambda(c + \sigma q + q - 1) < 1$ .

There remains consequently to consider the sum  $I_\sigma$  under the following conditions:  $\lambda(c + \sigma q) < 1 \leq \lambda(c + \sigma q + q - 1) < 2$ . We have

$$\frac{1}{2} <$$

$$\frac{1}{2} \leq \frac{1}{q} \sum_{x=c+\sigma q}^{c+\sigma q+q-1} \lambda(x) < \frac{3}{2}.$$

Reducing as in case ( $\alpha$ ) the sum  $I_\sigma$  to the form

$$I_\sigma = \sum_{r=0}^{q-1} \left\{ \frac{r + \nu(r)}{q} \right\},$$

we may write down the equation

$$\left\{ \frac{r + \nu(r)}{q} \right\} = \frac{r + \nu(r)}{q}$$

only when  $r = 0, 1, \dots, q-2$  (for now the case  $1 \leq \nu(r) < 2$  is possible), but when  $r = q-1$  this equation must be replaced by

$$\left\{ \frac{r + \nu(r)}{q} \right\} = \frac{r + \nu(r)}{q} - \delta,$$

where  $\delta$  may be equal to 0 or 1. Thus we get

$$\frac{1}{q} \sum_{x=c+\sigma q}^{c+\sigma q+q-1} \lambda(x) - \delta$$

$$I_\sigma = \frac{1}{2} q - \frac{1}{2} + \frac{1}{q} \sum_{x=c+\sigma q}^{c+\sigma q+q-1} \lambda(x) - \delta = \frac{1}{2} q + \rho_\sigma; |\rho_\sigma| < 1.$$

Substituting this expression for  $I_\sigma$  and expression (2) for  $I_s$ ,  $s \leq \sigma$ , in the equation (1), the validity of the lemma becomes obvious.

(ii) Now putting  $q = 1$ , it is evident that

$$-\frac{1}{2} m \leq S - \frac{1}{2} m q \leq \frac{1}{2} m.$$

The lemma is thus completely proved.

LEMMA III. Let  $p$  be a prime number  $> 2$ ,  $\alpha$  an integer not divisible by  $p$ ,  $h$  a positive integral number  $< p$  and  $\beta_\alpha$  any integer which depends on  $\alpha$ . Further let

$$S_\alpha = \sum_{x=0}^{h-1} \left\{ \frac{\alpha x + \beta_\alpha}{p} \right\}; \quad L_\alpha = S_\alpha - \frac{1}{2} h;$$

then the sum  $\sum |L_\alpha|$  extended over all numbers of the set

$$(3) \quad 1, 2, \dots, p-1 \quad \left( \text{i.e., } \sum_{\alpha=1}^{p-1} |L_\alpha| \right)$$

\* Lemma III also holds for  $p=2$ , as may be verified directly.

is less than

$$T = \sum_{x=1}^h \sum_{y=1}^{px-1} \left( \frac{p}{xy} + 1 \right),$$

where for every  $x$  the summation for  $y$  extends only over the numbers prime to  $x$ .

Proof. As a first step let us consider any single sum  $S_a$ . Supposing in Lemma I that  $k = h$  we can then find two relatively prime numbers  $x_0, y_0$  which satisfy the conditions

$$\alpha x_0 \equiv y_0 \pmod{p}; \quad 0 < x_0 \leq h; \quad 0 < |y_0| < \frac{p}{h}.$$

Hence we find  $\alpha x_0 = y_0 + t_0 p$ , where  $t_0$  is an integer. Moreover

$$\frac{\alpha}{p} = \frac{t_0}{x_0} + \frac{y_0}{x_0 p} = \frac{t_0}{x_0} + \frac{\theta_0}{x_0 h}; \quad |\theta_0| < 1.$$

Supposing  $m = [hx_0^{-1}]$ ,  $h_1 = h - mx_0$ , we get

$$S_a = \sum_{x=0}^{mx_0-1} \left\{ \frac{\alpha x + \beta_a}{p} \right\} + S_a'; \quad S_a' = \sum_{x=mx_0}^{mx_0+h_1-1} \left\{ \frac{\alpha x + \beta_a}{p} \right\}; \quad 0 \leq h_1 < h.$$

Hence, applying Lemma II, we find

$$S_a = \frac{1}{2} mx_0 + \frac{1}{2} \rho(m+1) + S_a' = \frac{1}{2} (h - h_1) + \frac{1}{2} \rho_0 \left( \frac{h}{x_0} + 1 \right) + S_a';$$

$$|\rho_0| < 1.$$

Putting  $k_1 = h_1$  and applying to the sum  $S_a'$  the same treatment as used in the case of the sum  $S_a$ , we obtain

$$S_a' = \frac{1}{2} (h_1 - h_2) + \frac{1}{2} \rho_1 \left( \frac{h_1}{x_1} + 1 \right) + S_a''; \quad |\rho_1| < 1; \quad 0 \leq h_2 < h_1,$$

where the sum  $S_a''$  consists of  $h_2$  terms. In the same manner we find

$$S_a'' = \frac{1}{2} (h_2 - h_3) + \frac{1}{2} \rho_2 \left( \frac{h_2}{x_2} + 1 \right) + S_a'''; \quad |\rho_2| < 1; \quad 0 \leq h_3 < h_2,$$

and so on, until we reach some  $h_{n+1} = 0$ . Thus we find finally

$$S_a = \frac{1}{2} h + \frac{1}{2} \sigma \left[ \left( \frac{h}{x_0} + 1 \right) + \left( \frac{h_1}{x_1} + 1 \right) + \cdots + \left( \frac{h_n}{x_n} + 1 \right) \right]; \quad |\sigma| < 1.$$

The lemma will be proved if we can show that

$$\Omega = \frac{1}{2} \sum_{\alpha} \left[ \left( \frac{h}{x_0} + 1 \right) + \left( \frac{h_1}{x_1} + 1 \right) + \cdots + \left( \frac{h_n}{x_n} + 1 \right) \right] < T,$$

where the summation extends over all numbers of the set (3). It is necessary to notice that the number  $n$ , as well as the numbers  $h_1, h_2, \dots, h_n, x_0, x_1, \dots, x_n$ , depends on the value attributed to  $\alpha$ , and for a given  $\alpha$  the numbers  $x_0, x_1, \dots, x_n$  are different. In order to estimate the sum  $\Omega$  we shall first determine an upper bound of the sum of those terms  $k/x+1$  which correspond to the same value of  $x$ . The given  $x$  can correspond only to those values of  $\alpha$  which satisfy the congruence  $\alpha x \equiv y \pmod{p}$ , where  $y$  is an integer prime to  $x$  and  $|y| < pk^{-1}$  and therefore also  $|y| < px^{-1}$ . Hence for a given  $x, y$  can take only the values  $\pm 1, \pm 2, \dots, \pm [px^{-1}]$  prime to  $x$ . For every such  $y$  we shall find a corresponding value of  $\alpha$ . To every admissible system of numbers  $x, y, \alpha$  corresponds some  $k$ , which satisfies the condition  $|y| < pk^{-1}$ , or  $k < p|y|^{-1}$ . Therefore the sum of all the terms in the sum  $\Omega$  which correspond to a given  $x$  will be less than

$$\sum_{y=1}^{px^{-1}} \left( \frac{p}{xy} + 1 \right),$$

where  $y$  ranges over numbers prime to  $x$ . From this Lemma III follows immediately.

non-negative

LEMMA IV. Let  $p$  be a prime number  $> 2^*$ ,  $\beta$  or  $\beta_\alpha$  an integer which may depend on  $\alpha$ , and  $h$  and  $\gamma$  integers which satisfy the conditions  $0 < h < p$ ;  $0 < \gamma < p$ . Let us denote by the symbol  $R_\alpha$  the number of least positive residues of

$$\alpha x + \beta_\alpha \quad (x=0, 1, \dots, h-1)$$

which are less than a given number  $\gamma$ , and let us suppose

$$R_\alpha = h\gamma p^{-1} + H_\alpha;$$

then extending the summation over all  $\alpha = 1, 2, \dots, p-1$  we shall get

$$\sum |H_\alpha| < 2T.$$

Proof. According to Lemma III and putting

$$S'_\alpha = \sum_{x=0}^{h-1} \left\{ \frac{\alpha x + \beta - \gamma}{p} \right\} = \frac{1}{2} h + L'_\alpha; \quad S_\alpha = \sum_{x=0}^{h-1} \left\{ \frac{\alpha x + \beta}{p} \right\} = \frac{1}{2} h + L_\alpha,$$

we have

$$\sum |L'_\alpha| < T; \quad \sum |L_\alpha| < T; \quad \sum |S'_\alpha - S_\alpha| < 2T.$$

\*It can be verified directly that Lemma IV holds for  $p=2$  also.

It is easy to see that

$$(i) \text{ if } \left\{ \frac{\alpha x + \beta}{p} \right\} < \frac{\gamma}{p},$$

then

$$\left\{ \frac{\alpha x + \beta - \gamma}{p} \right\} = \left\{ \frac{\alpha x + \beta}{p} \right\} + 1 - \frac{\gamma}{p};$$

(ii) if

$$\left\{ \frac{\alpha x + \beta}{p} \right\} \geq \frac{\gamma}{p},$$

then

$$\left\{ \frac{\alpha x + \beta - \gamma}{p} \right\} = \left\{ \frac{\alpha x + \beta}{p} \right\} - \frac{\gamma}{p}.$$

Therefore

$$S'_a - S_a = R_a - \frac{h\gamma}{p} = H_a,$$

which proves the lemma since  $\sum |S'_a - S_a| < 2T$ .

**THEOREM.** Let  $p$  be a prime number  $> 2$ ,  $e$  a factor of  $p - 1$ ,  $a$  an integer not divisible by  $p$  and  $b$  any given integer. Distributing all the numbers  $1, 2, \dots, p - 1$  into  $e$  classes and referring to the  $i$ th class all those, the indices of which are congruent to  $i \pmod{e}$ , the number of numbers of any class, which belong  $\pmod{p}$  to an arithmetical progression  $ax + b$ ;  $x = 0, 1, \dots, h - 1$  ( $0 < h < p$ ) can be represented in the form

$$\frac{h}{e} + \Delta; \Delta^2 < T + \frac{1}{2}p.$$

**Proof.** Let  $(p - 1)e^{-1} = f$  and let us consider a set of  $fh$  numbers of the form

$$(4) \quad \alpha(ax + b),$$

where  $\alpha$  ranges over all the numbers of the  $i$ th class, while  $x$ , independently of  $\alpha$ , ranges over all the numbers  $0, 1, \dots, h - 1$ . To every number of the set (4) we can find one and only one number  $u$ , which satisfies the conditions

$$au + b \equiv \alpha(ax + b) \pmod{p}; \quad 0 \leq u < p,$$

$$0 \leq u < p$$

and where the number  $u$ , after introduction of  $a'$  by means of the congruence  $aa' \equiv 1 \pmod{p}$ , can be determined by the following conditions:

$$(5) \quad u \equiv \alpha x + \beta_a \pmod{p}; \quad \beta_a = aba' - ba'; \quad 0 \leq u < p.$$

\* The theorem can be verified directly for  $p=2$ .

Let  $D$  be the number of numbers  $u$ , which are  $< h$ , obtained in this way.\* The idea of the following proof consists in evaluating the number  $D$  by two different methods.

(i) If we leave  $\alpha$  constant, then  $\beta_\alpha$  also does not vary, and therefore, in view of congruence (5), the number of values of  $u$  less than  $h$ , which correspond to all the numbers of the set

$$\alpha(ax + b) \quad (x = 0, 1, \dots, h-1)$$

may be represented in accordance with Lemma IV in the form

$$\frac{h^2}{p} + H_\alpha,$$

where on extending the summation not only over numbers  $\alpha$  of the  $i$ th class, but over all  $\alpha = 1, 2, \dots, p-1$ , we shall obtain

$$(6) \quad \sum |H_\alpha| < 2T;$$

and since the number of numbers  $\alpha$  of the  $i$ th class is  $f$ ,

$$D = f \frac{h^2}{p} + \sum_i H_\alpha,$$

where  $\sum_i$  denotes the sum extended over all numbers of the  $i$ th class.

(ii) Let there be in the set

$$(7) \quad ax + b \quad (x = 0, 1, \dots, h-1)$$

$c_0$  numbers of class 0,  $c_1$  numbers of class 1,  $\dots$ ,  $c_{e-1}$  numbers of class  $e-1$ . The symbol  $c_s$  we shall later use also, when  $s \geq e$ , denoting by it the number of numbers of the class, the index of which is the lowest positive residue of number  $s$ , modulo  $e$ . Multiplying one of the numbers of the  $j$ th class of the set (7) by all numbers of the  $i$ th class, and putting instead of these products the numbers  $au+b$ ,  $0 \leq u < p$ , congruent to them modulo  $p$ , we shall obtain  $f$  numbers  $au+b$  which evidently belong to the class  $i+j$ . Among these numbers  $au+b$  there will obviously be  $c_{i+j}$  numbers for which  $u < h$ . Therefore taking into consideration that  $j$  can take only the values  $0, 1, \dots, e-1$  we find

$$D = c_0 c_i + c_1 c_{i+1} + c_2 c_{i+2} + \dots + c_{e-1} c_{i+e-1}. \quad **$$

Comparing this value of  $D$  with that obtained before, we get

(see \*\*).

$$c_0 c_i + c_1 c_{i+1} + \dots + c_{e-1} c_{i+e-1} = f \frac{h^2}{p} + \sum_i H_\alpha.$$

\* This definition of  $D$  is incorrect for the subsequent work. What the author really means is to let  $D$  be the number of pairs  $\langle \alpha, x \rangle$  such that  $(\text{index } \alpha) \equiv i \pmod{e}$ ,  $0 \leq x \leq h-1$ , and  $0 \leq u < h$ , where  $u$  satisfies (5). In (i),  $D$  is calculated by fixing  $\alpha$ , finding the number of corresponding  $x$ , and then summing over  $\alpha$ . In (ii), the same procedure is followed with  $\alpha$  and  $x$  interchanged. (Here  $\langle \alpha, x \rangle = \langle \alpha', x' \rangle$  if and only if  $\alpha = \alpha'$  and  $x = x'$ .)

\*\* This should read  $D = c_0 c_i + c_1 c_{i+1} + \dots + c_{e-1} c_{i+e-1} + T_0$ , where  $T_0 = f$  if there is an  $x$ ,  $0 \leq x \leq h-1$ , with  $ax+b \equiv 0 \pmod{p}$ , and  $T_0 = 0$  otherwise.

original  $i$ th  
class

Let

$$c_s = \frac{h}{e} + \delta_s;$$

then, since  $c_0 + c_1 + \dots + c_{e-1}$  may be represented in the form  $h - \sigma$ ,  $\sigma = 0$ , or 1 (because one of the numbers (7) may be divisible by  $p$ ), we shall have  $\delta_0 + \delta_1 + \dots + \delta_{e-1} = -\sigma$ ; whence we obtain

$$(8) \quad \frac{h^2}{e} - \frac{2h\sigma}{e} + \delta_0\delta_1 + \delta_1\delta_{i+1} + \dots + \delta_{e-1}\delta_{i+e-1} = f \frac{h^2}{p} + \sum_i H_\alpha.$$

+  $\sum_i H_\alpha - \tau_0$   
where  $\tau_0 = 0$  or  $f$ ,  
according as  
 $\sigma = 0$  or  $1$ .

From this, extending the summation over all  $i = 1, 2, \dots, e-1$ , we find

$$\frac{h^2(e-1)}{e} - 2h\sigma \frac{e-1}{e} + \sigma^2 - \delta_0^2 - \delta_1^2 - \dots - \delta_{e-1}^2$$

$$= \frac{f(e-1)h^2}{p} + \sum_{i=1}^{e-1} \sum_i H_\alpha,$$

$$+ \sum_{i=1}^{e-1} \sum_i H_\alpha - \tau(e-1)$$

and hence

$$\sum_{s=0}^{e-1} \delta_s^2 < \sum_{i=1}^{e-1} |\sum_i H_\alpha| + p \frac{e-1}{e}.$$

$$\sum_{s=0}^{e-1} \delta_s^2 \leq$$

Also putting  $i=0$  in (8) we get

$$\sum_{s=0}^{e-1} \delta_s^2 < |\sum_0 H_\alpha| + \frac{p}{e}.$$

Adding the two last inequalities, and dividing by 2, we obtain

$$\sum_{s=0}^{e-1} \delta_s^2 < T + \frac{1}{2} p$$

and, in particular, for each  $r$

$$\delta_r^2 < T + \frac{1}{2} p; \left(c_r - \frac{h}{e}\right)^2 < T + \frac{1}{2} p,$$

which proves the theorem.

Note. Evident transformations give an upper bound of  $|\Delta|$  less than  $\sqrt{p} \log p$ .

LENINGRAD, RUSSIA

# ON THE BOUND OF THE LEAST NON-RESIDUE OF $n$ th POWERS\*

BY

J. M. VINOGRADOV

1. In my paper *On the distribution of residues and non-residues of powers* (Journal of the Physico-Mathematical Society of Perm, 1919) I demonstrated that the least quadratic non-residue of a prime  $p$  is less than

$$p^{1/2}(\log p)^2$$

for all sufficiently great values of  $p$ .

Using the same method one can establish a more general theorem:

**THEOREM I.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  distinct from 1, the least non-residue of  $n$ th powers modulo  $p$  is less than*

$$p^{1/2k}(\log p)^2; \quad k = e^{(n-1)/n}$$

for all sufficiently great values of  $p$ .

This bound may be considerably lowered, by means of very simple changes in our method. For example one can demonstrate the following theorems:

**THEOREM II.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than 20, the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/6}$  for all sufficiently great values of  $p$ .*

**THEOREM III.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than 204, the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/8}$  for all sufficiently great values of  $p$ .*

We prove finally the general theorem:

**THEOREM IV.** *If  $p$  is a prime and  $n$  a divisor of the number  $p-1$  greater than  $m^m$ , where  $m$  is an integer  $\geq 8$ , the least non-residue of  $n$ th powers modulo  $p$  is less than  $p^{1/m}$  for all sufficiently great values of  $p$ .*

2. First we shall demonstrate Theorem I. We use the notations

$$P = p^{1/2}(\log p)^2; \quad T = p^{1/2k}(\log p)^2; \quad k = e^{(n-1)/n},$$

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\*Presented to the Society, September 9, 1926; received by the editors in January, 1926.

and assume that there are no non-residues of  $n$ th powers modulo  $p$  less than  $T$ . Then only numbers divisible by integers greater than  $T$  and less than  $P$  can be non-residues of  $n$ th powers less than  $P$ . But evidently, of such numbers, there are not more than

$$\sum_{\substack{q < P \\ q > T}} \left[ \frac{P}{q} \right],$$

where  $q$  runs only over primes. Using the known law of distribution of primes, we may bring this expression to the form

$$\begin{aligned} P \log \frac{\log P}{\log T} + O\left(\frac{P}{\log p}\right) &= P \left[ \frac{n-1}{n} + \log \frac{1 + \frac{4 \log \log p}{\log p}}{1 + \frac{4k \log \log p}{\log p}} \right] + O\left(\frac{P}{\log p}\right) \\ &= \left( \frac{n-1}{n} + \frac{(4-4k) \log \log p}{\log p} \right) P + O\left(\frac{P}{\log p}\right). \end{aligned}$$

On the other hand, according to my previous work, the number of residues of  $n$ th powers modulo  $p$  in the range

$$1, 2, \dots, [P]$$

may be given as follows:

$$\frac{[P]}{n} + \Delta; \quad |\Delta| < p^{1/2} \log p.$$

Thus the number of non-residues in the same range may be expressed by the formula

$$P\left(\frac{n-1}{n}\right) + \rho; \quad |\rho| < p^{1/2} \log p + 1.$$

Hence

$$P\left(\frac{n-1}{n}\right) + \rho \leq P\left(\frac{n-1}{n} + \frac{(4-4k) \log \log p}{\log p}\right) + O\left(\frac{P}{\log p}\right)$$

which brings us to the inequality

$$(4k-4) \log \log p \leq O(1),$$

which is impossible for sufficiently great  $p$ . This proves Theorem I.

3. To prove Theorem II, let

$$P = p^{1/2}(\log p)^2; T = p^{1/6},$$

and assume that there are no non-residues of  $n$ th powers modulo  $p$  less than  $T$ . Then only numbers divisible by primes greater than  $T$  and less than  $P$  can be non-residues less than  $P$ . The number of such numbers is evidently equal to

$$(1) \quad \sum_{q > T}^q \left[ \frac{P}{p} \right] - \sum_{q > T}^{q < P^{1/2}} \sum_{q_1 > q}^{q_1 < P/q} \left[ \frac{P}{qq_1} \right] + \sum_{q > T}^{q < P^{1/2}} \sum_{q_1 > q}^{q_1 < (P/q)^{1/2}} \sum_{q_2 > q_1}^{q_2 < P/q_1} \left[ \frac{P}{qq_1q_2} \right],$$

where  $q, q_1, q_2$  run over primes.

But, according to the law of the distribution of primes, the first sum may be written

$$P \log \frac{\log P}{\log T} + O\left(\frac{P}{\log p}\right) = P \log 3 + O\left(\frac{P \log \log p}{\log p}\right),$$

which for sufficiently great  $p$  is less than

$$P \cdot 1.0987.$$

The second double sum may be put into the form

$$P \sum_{q > T}^{q < P^{1/2}} \frac{1}{q} \log \frac{\log (P/q)}{\log p} + O\left(\frac{P}{\log p}\right) = P \sum_{q > p^{1/6}}^{q < p^{1/2}} \frac{1}{q} \log \frac{\log p^{1/2}}{\log q} + O\left(\frac{P \log \log p}{\log p}\right).$$

But applying the law of distribution of primes we have

$$\begin{aligned} & P \int_{p^{1/6}}^{p^{1/2}} \log \frac{\log (p^{1/2}/z)}{\log z} \cdot \frac{dz}{z \log z} + O\left(\frac{P \log \log p}{\log p}\right) \\ &= P \int_{1/3}^{1/2} \log \frac{1-u}{u} \cdot \frac{du}{u} + O\left(\frac{P \log \log p}{\log p}\right), \end{aligned}$$

which, for  $p$  sufficiently great, is greater than

$$P \cdot 0.147.$$

The last triple sum evidently is a quantity of the order

$$P \frac{\log \log p}{\log p},$$

so that the expression (1) for sufficiently great  $p$  is less than

$$P(1.0988 - 0.147) = P \cdot 0.9518.$$

On the other hand, the number of non-residues of  $n$ th powers modulo  $p$  in the series

$$1, 2, \dots, [P],$$

as seen in § 2, is equal to

$$P \left(1 - \frac{1}{n}\right) + O\left(\frac{P}{\log p}\right).$$

So, for  $p$  sufficiently great, we have the inequality

$$P \left(1 - \frac{1}{n}\right) < P \cdot 0.952.$$

The impossibility of this inequality for  $n > 20$  proves Theorem II.

4. To prove Theorem III we let

$$P = p^{1/2}(\log p)^2; \quad T = p^{1/8},$$

and assume that there are no non-residues of  $n$ th powers, modulo  $p$ , less than  $T$ . It is easy to show that the number of such numbers is less than

$$(2) \quad \sum_{q>T}^{q \leq P} \left[ \frac{P}{q} \right] - \sum_{q>T}^{q \leq P^{1/2}} \sum_{q_1>q}^{q_1 \leq P/q} \left[ \frac{P}{qq_1} \right] + \sum_{q>T}^{q \leq P^{1/2}} \sum_{q_1>q}^{q_1 \leq (P/q)^{1/2}} \sum_{q_2>q_1}^{q_2 \leq P/q q_1} \left[ \frac{P}{qq_1 q_2} \right],$$

where  $q, q_1, q_2$  run over primes only.

Applying the known laws of distribution of primes, we can put this expression into the form

$$\sum_{q>p^{1/8}}^{q \leq p^{1/2}} \frac{P}{q} - \sum_{q>p^{1/8}}^{q \leq p^{1/4}} \sum_{q_1>q}^{q_1 \leq p^{1/2}/q} \frac{P}{qq_1} + \sum_{q>p^{1/8}}^{q \leq p^{1/8}} \sum_{q_1>q}^{q_1 \leq p^{1/4}/q^{1/2}} \sum_{q_2>q}^{q_2 \leq p^{1/2}/q q_1} \frac{P}{qq_1 q_2} \\ + O\left(\frac{P \log \log p}{\log p}\right).$$

The first sum may be put into the form

$$P \log 4 + O\left(\frac{P}{\log p}\right)$$

which for sufficiently great  $p$  is less than

$$P \cdot 1.3863.$$

Then as in the proof of Theorem II the second double sum may be given in the form

$$P \int_{1/4}^{1/2} \log \frac{1-u}{u} \frac{du}{u} + O\left(\frac{P}{\log p}\right),$$

which for sufficiently great  $p$  is less than

$$P \cdot 0.40609.$$

It remains to estimate the third triple sum. We have

$$\sum_{\substack{q_1 < p^{1/2}/q q_1 \\ q_2 > q_1}} \frac{P}{qq_1 q_2} = \frac{P}{qq_1} \log \frac{\frac{1}{2} \log p - \log q - \log q_1}{\log q_1} + O\left(\frac{P}{qq_1 \log p}\right).$$

Noting this, it is easy to obtain

$$\begin{aligned} \sum_{\substack{q_1 < p^{1/4} q^{1/2} \\ q_2 > q_1}} \sum_{\substack{q_1 < p^{1/2}/q q_1 \\ q_2 > q_1}} \frac{P}{qq_1 q_2} &= \frac{P}{q} \int_q^{p^{1/4}/q^{1/2}} \frac{dy}{y \log y} \cdot \log \frac{\frac{1}{2} \log p - \log q - \log y}{\log y} \\ &+ O\left(\frac{P}{q \log p}\right) = \frac{P}{q} \int_v^{1/4-v/2} \frac{dz}{z} \log \frac{\frac{1}{2} - v - z}{z} + O\left(\frac{P}{q \log p}\right); \quad v = \frac{\log q}{\log p}. \end{aligned}$$

The third triple sum may be given in the form

$$\begin{aligned} P \int_{1/8}^{1/6} \frac{dv}{v} \int_v^{1/4-v/2} \frac{dz}{z} &\left( \log\left(\frac{1}{2} - v\right) - \log z - \frac{z}{\frac{1}{2} - v} - \frac{z^2}{2(\frac{1}{2} - v)^2} \right. \\ &\quad \left. - \frac{z^3}{3(\frac{1}{2} - v)^3} - \dots \right) + O\left(\frac{P}{\log p}\right) \\ &= P \int_{1/8}^{1/6} \log \frac{\frac{1}{2}(\frac{1}{2} - v)}{v} \log \left(\frac{2(\frac{1}{2} - v)}{v}\right)^{1/2} \frac{dv}{v} \\ &\quad - P \int_{1/8}^{1/6} \left(\frac{1}{2} + \frac{1}{4 \cdot 4} + \frac{1}{8 \cdot 9} + \frac{1}{16 \cdot 16} + \dots\right) \frac{dv}{v} \\ &\quad + P \int_{1/8}^{1/6} \left(\frac{v}{\frac{1}{2} - v} + \frac{1}{4} \left(\frac{v}{\frac{1}{2} - v}\right)^2 + \frac{1}{9} \left(\frac{v}{\frac{1}{2} - v}\right)^3 + \dots\right) \frac{dv}{v}. \end{aligned}$$

Introducing in the first integral the substitution

$$\frac{\frac{1}{2} - v}{v} = u,$$

and in the third the substitution

$$\frac{v}{\frac{1}{2} - v} = u,$$

we easily obtain

$$P \int_2^3 \log \frac{u}{2} \log 2u^{1/2} \frac{du}{1+u} - P \left( \frac{1}{2} + \frac{1}{4 \cdot 4} + \frac{1}{8 \cdot 9} + \cdots \right) \log \frac{4}{3} \\ + P \int_{1/3}^{1/2} \left( 1 + \frac{1}{4}u + \frac{1}{9}u^2 + \cdots \right) \frac{du}{1+u} + O\left(\frac{P}{\log p}\right).$$

But this expression for sufficiently great  $p$  is less than

$$P \cdot 0.01489.$$

Comparing this result with those obtained for simple and double sums we find that the expression (2) for sufficiently great  $p$  is less than

$$P(1.38631 - 0.40609 + 0.01489) < P\left(1 - \frac{1}{205}\right),$$

whence, reasoning as in Theorem II, we prove Theorem III.

5. Passing to the demonstration of Theorem IV let us prove first the following lemma:

LEMMA. *If  $k$  be a positive number increasing indefinitely, and  $s$  an integer  $\geq 2$ , then the number  $T$  of numbers less than  $t_s$  and not divisible by primes greater than  $k$ , where  $t_s$  is any number satisfying the condition*

$$k^s < t_s \leq k^{s+1/(s+2)},$$

*is greater than*

$$\frac{t_s}{s!(s+2)^s}$$

*for all sufficiently great values of  $k$ .*

Demonstration. Let

$$\epsilon = \frac{1}{s+2}.$$

(i) Taking any number  $t_1$  such that

$$k < t_1 < k^{2-2\epsilon},$$

we find a lower bound of the number  $T_1$  of numbers which are  $\leq t_1$  and divisible at least by one prime greater than  $k^{1-\epsilon}$  and  $\leq k$ . Evidently

$$T_1 = \sum_{\substack{q \leq k \\ q > k^{1-\epsilon}}} \left[ \frac{t_1}{q} \right],$$

where  $q$  runs over primes only. Considering certain laws of distribution of primes, this number may be written in the form

$$t_1 \log \frac{\log t_1}{(1-\epsilon) \log k} + O\left(\frac{t_1}{\log k}\right).$$

But this last expression is greater than

$$t_1 \log \frac{1}{1-\epsilon} + O\left(\frac{t_1}{\log k}\right)$$

which for sufficiently great  $k$  is greater than  $\epsilon t_1$ .

So for sufficiently great  $k$  we have

$$T_1 > \epsilon t_1.$$

(ii) Taking any number  $t_2$ ,

$$k^2 < t_2 \leq k^{3-3\epsilon},$$

we find a lower bound of the number  $T_2$  of numbers which are  $\leq t_2$  and divisible by the product of any two primes, greater than  $k^{1-\epsilon}$  and  $\leq k$ . Products differing in the order of divisors, we shall consider as different.

Let  $q$  be a prime greater than  $k^{1-\epsilon}$  and  $\leq k$ . The numbers not surpassing  $t_2$  and divisible by  $q$  are

$$q, 2q, \dots, \left[ \frac{t_2}{q} \right] q.$$

Consequently, we must find how many numbers of the series

$$1, 2, \dots, \left[ \frac{t_2}{q} \right]$$

are still divisible by primes greater than  $k^{1-\epsilon}$  and  $\leq k$ . Since

$$k = k^{2-1} < \frac{t_2}{q} < k^{3-3\epsilon-(1-\epsilon)} = k^{2-2\epsilon},$$

then, according to (i), we find that this number for sufficiently great  $k$  is greater than

$$\epsilon \frac{t_2}{q}.$$

Hence, as in (i), we find that

$$T_2 > \epsilon^2 t_2$$

for all sufficiently great values of  $k$ .

(iii) Arguing thus, we finally find that, if  $t_s$  is any number satisfying the condition

$$k^s < t_s \leq k^{s+1-(s+1)\epsilon},$$

and  $T_s$  denotes the number of numbers  $\leq t_s$  and divisible by the product of  $s$  primes greater than  $k^{1-\epsilon}$  and  $\leq k$  (considering as different the products with different order of divisors), then for sufficiently great  $k$

$$T_s > \epsilon^s t_s = \frac{t_s}{(s+2)^s}.$$

Noting that

$$T > \frac{T_s}{s!},$$

we prove the lemma.

**Demonstration of Theorem IV.** We have seen that, if  $n$  is a divisor of  $p-1$  differing from 1, the number  $R$  of residues of  $n$ th powers modulo  $p$  less than  $p^{1/2}(\log p)^2$  can be written in the form

$$(3) \quad R = \frac{p^{1/2}(\log p)^2}{n} + O(p^{1/2} \log p).$$

Taking any integer  $m \geq 8$ , and letting  $k = p^{1/m}$ ;  $s = m/2$  for  $m$  even;  $s = (m+1)/2$  for  $m$  odd, according to the lemma the number of numbers less than  $p^{1/2}(\log p)^2$ , divisible only by primes less than  $p^{1/m}$ , is for  $p$  sufficiently great, greater than

$$\frac{p^{1/2}(\log p)^2}{s!(s+2)^s}.$$

Assuming that among the numbers less than  $p^{1/m}$  there are no non-residues of  $n$ th powers modulo  $p$ , we have

$$R > \frac{p^{1/2}(\log p)^2}{s!(s+2)^s}.$$

Comparing this inequality with equation (3) we have

$(1/n) + O(1/\log p) > 1/(s!(s+2)^s)$  whence  $n < s!(s+2)^s + \delta$ , where  $\delta$  goes to 0 with increasing  $p$ . But applying the formula of Stirling, we have  $s!(s+2)^s < m^m$ , from which it follows that, for sufficiently great values of  $p$ ,  $n < m^m$ , which is impossible for  $n > m^m$ . This proves the Theorem IV.

**Remark.** Evidently the bound  $n > m^m$  is very rough. Thus, with  $m=8$ , we get here the inequality  $n > 16777216$  instead of the inequality  $n > 204$  found above.

6. We know that to find a primitive root of a prime  $p$  it is enough, having found different primitive divisors  $2, q_1, q_2, \dots, q_r$  of the number  $p-1$ , to find one further non-residue  $\nu_0, \nu_1, \dots, \nu_r$  of each of the powers  $2, q_1, \dots, q_r$ . By means of the numbers  $\nu_0, \nu_1, \dots, \nu_r$  it is quite easy to find the primitive root. Applying the established theorems it is easy to prove that

(i) If  $p$  is sufficiently great, all the numbers  $\nu_0, \nu_1, \dots, \nu_r$  are found in the range

$$(4) \quad 1, 2, \dots, [p^{1/2e^{1/2}}(\log p)^2].$$

(ii) If  $p$  is not of the form  $8N+1$ , and the numbers  $q_1, q_2, \dots, q_r$  are sufficiently large, then instead of the range (4) we can take shorter ranges, depending on the lowest bound  $Q$  of the numbers  $q$ . For example, if  $Q > 20$ , we take the range

$$(5) \quad -1, 1, 2, \dots, [p^{1/6}];$$

if  $Q > 204$ , then

$$(6) \quad -1, 1, 2, \dots, [p^{1/8}],$$

and finally if  $Q > m^m$ , when  $m$  is an integer  $\geq 8$ ,

$$(7) \quad -1, 1, 2, \dots, [p^{1/m}].$$

These results can be formulated in a different manner.

(i) If  $p$  is a sufficiently great prime, then a complete system of residues modulo  $p$  can be got by multiplying the powers of the numbers of the range (4).

(ii) If  $p$  is not of the form  $8N+1$ , and all the numbers  $q_1, q_2, \dots, q_r$  are not less than  $Q$ , then instead of the range (4) we can take the range (5) for  $Q > 20$ , the range (6) for  $Q > 204$ , and finally the range (7) for  $Q = m^m; m \geq 8$ .

LENINGRAD, RUSSIA

## ON CONVERGENCE FACTORS IN MULTIPLE SERIES\*

BY  
CHARLES N. MOORE

1. **Introduction.** Convergence factors may be defined as a set of functions of one or more parameters which, when introduced as factors of the terms of a series, cause a divergent series to converge or a series which is already convergent to converge more rapidly throughout a given range of values of the parameters. In the case of the convergence factors generally used in practice it is further true that each factor approaches unity as the parameters approach certain limit-values. Furthermore the function defined by the series for the given range of values of the parameters approaches a definite limit as the parameters approach the limit-values, this limit being the value of the series for convergent series and a value we find it useful to ascribe to the series in the case of a divergent series. In the following discussion we shall find it convenient to distinguish between the cases where convergence factors merely have the property of reducing a series to convergence for a certain range of values of their parameters and the case where they have the additional property of obtaining a value for the series by the process of allowing the parameters to approach certain limit-values. We will refer to convergence factors that are only known to have the first property as being of type I, and convergence factors that have both properties as being of type II.

Several sets of sufficient conditions for convergence factors of type II in single series that are summable by Cesàro's method have been obtained by various writers.† The most general of these sets is that due to Bromwich. Hurwitz has obtained a set of necessary and sufficient conditions for this case,‡ which are quite similar in form to Bromwich's conditions,

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†Cf. for example L. Fejér, *Mathematische Annalen*, vol. 58 (1904), p. 51; G. H. Hardy, *Proceedings of the London Mathematical Society*, (2), vol. 4 (1906), p. 247; *Mathematische Annalen*, vol. 64 (1907), p. 77; C. N. Moore, *these Transactions*, vol. 8 (1907), p. 299; T. J. Bromwich, *Mathematische Annalen*, vol. 65 (1908), p. 350; S. Chapman, *Proceedings of the London Mathematical Society*, (2), vol. 9 (1911), p. 369.

‡Cf. abstract, *Bulletin of the American Mathematical Society*, vol. 25 (1922), p. 156. For a statement of Hurwitz's conditions in the case of integral orders of summability see D. S. Morse, *Relative inclusiveness of certain definitions of summability*, *American Journal of Mathematics*, vol. 45 (1923), pp. 263, 264.

only a slight increase of generality in one of these being needed in order to make them both necessary and sufficient. Necessary and sufficient conditions for convergence factors of type I in single series have been given by Kojima.\* These conditions, as might be expected, may be obtained from Hurwitz's conditions by dropping the requirement that the convergence factors approach unity as the parameter approaches a certain limit-value. This condition is obviously without significance in the case of convergence factors of type I.

Sufficient conditions for convergence factors of type I in double series summable  $(Ck)$ , where  $k$  is a positive integer or zero, have been given by the present writer,† and an analogous theorem for triple series summable  $(C1)$  has been obtained by Bess M. Eversull.‡ The discussion of convergence factors of type II in double series in my paper of 1913 contains an error, and as a result the conditions there given§ are not in themselves sufficient. They can be made so by the addition of conditions  $(D_1)$  and  $(D_2)$  of Theorem III of this paper.

The purpose of the present paper is to obtain necessary and sufficient conditions for convergence factors of both types in the case of multiple series that are summable  $(Ck)$ , where  $k$  is any positive integer or zero. The theorems will be stated both for double series and for multiple series of order  $n$ . For the sake of simplicity in writing, the proofs will be carried through for the case of double series, the proper changes for the analogous proofs in the case of multiple series of higher order being indicated where this seems desirable.||

For the applications of convergence factor theorems that arise in problems in mathematical physics, most of the various sets of sufficient conditions that have been given are adequate. For such applications, however, as the study of the general theory of series and the relationship between various methods for summing divergent series it seems highly important to have sets of conditions that are both necessary and sufficient.

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\*Kojima, *On generalized Toeplitz's theorems on limit and their applications*, Tôhoku Mathematical Journal, vol. 12 (1917), pp. 291-326. See in particular §§7, 8. References to previous sets of sufficient conditions for this case may be found in this paper.

†Cf. *On convergence factors in double series and the double Fourier's series*, these Transactions, vol. 14 (1913), pp. 73-104. See in particular Lemma 4.

‡Cf. Lemma 8 of her paper, *On convergence factors in triple series and the triple Fourier's series*, Annals of Mathematics, (2), vol. 24 (1922-23). For sufficient conditions for convergence factors of type II in triple series summable  $(C1)$  see Theorem II of this paper.

§Loc. cit., Theorem III; and for the special case of series summable  $(C1)$ , Theorem II.

||The reader will of course readily see how to adapt the proof to the case of single series.

2. **Definitions and fundamental identities.** The general term in a multiple series of order  $n$  may be identified by a set of subscripts  $(i_1, i_2, \dots, i_n)$ . By way of abbreviation, we shall use the symbol  $[i]$  to denote this set of subscripts. The series may then be represented by the notation  $\sum a_{[i]}$ . Using  $[m]$  similarly in place of the set of symbols  $(m_1, m_2, \dots, m_n)$ , we set

$$(1) \quad \begin{aligned} S_{[m]} &= \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} a_{[i]}, \\ S_{[m]}^{(k)} &= \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \frac{\Gamma(k+m_1-i_1)}{\Gamma(k)\Gamma(m_1-i_1+1)} \cdot \frac{\Gamma(k+m_2-i_2)}{\Gamma(k)\Gamma(m_2-i_2+1)} \\ &\quad \cdots \frac{\Gamma(k+m_n-i_n)}{\Gamma(k)\Gamma(m_n-i_n+1)} S_{[i]}, \end{aligned}$$

$$(2) \quad A_{[m]}^{(k)} = \frac{\Gamma(m_1+k)}{\Gamma(k+1)\Gamma(m_1)} \cdot \frac{\Gamma(m_2+k)}{\Gamma(k+1)\Gamma(m_2)} \cdots \frac{\Gamma(m_n+k)}{\Gamma(k+1)\Gamma(m_n)}.$$

If the quotient  $(S_{[m]}^{(k)}/A_{[m]}^{(k)})$  approaches a limit  $S$  as  $m_1, m_2, \dots, m_n$  become infinite, we say that the series is summable  $(Ck)^*$  to the value  $S$ .

Corresponding to the  $r$ th differences of single sequences, we introduce the following definition of the  $r$ th difference of a multiple sequence of order  $n$ :

$$(3) \quad \Delta_{rr \dots r} f_{[i]} = \sum_{s_1=0}^r \cdots \sum_{s_n=0}^r (-1)^{s_1} \cdots (-1)^{s_n} \binom{r}{r-s_1} \cdots \binom{r}{r-s_n} f_{[i+s]},$$

where it is understood that the subscript  $r$  is repeated  $n$  times, and that  $[i+s]$  stands for the set of numbers  $i_1+s_1, i_2+s_2, \dots, i_n+s_n$ . For differences of this type with certain terms missing we use the notation† defined by

$$(4) \quad \Delta_{(r,p_1)(r,p_2) \dots (r,p_n)} f_{[i]} = \sum_{s_1=0}^{r-p_1} \cdots \sum_{s_n=0}^{r-p_n} (-1)^{s_1} \cdots (-1)^{s_n} \binom{r}{r-s_1} \cdots \binom{r}{r-s_n} f_{[i+s]}.$$

\*The above definition may be used for any value of  $k$  except negative integers. In this paper we shall confine ourselves to the case where  $k$  is a positive integer or zero. It is apparent that the definition could be generalized even for this case by introducing  $n$  different indices instead of the single index  $k$ . The appropriate changes to make in the various convergence factor theorems in order to have them apply to this more general case are fairly obvious.

†For the sake of simplicity in writing, a subscript of  $\Delta$  of the form  $(r, r)$  will be replaced by 0 in subsequent formulas.

In the case of double series there was obtained in my paper of 1913 a certain fundamental identity,\* involving the terms of the series with convergence factors, the expressions (1), and the differences (3) and (4). In our present notation this identity may be written†

$$(5) \quad \sum_{i=1}^{p+r} \sum_{j=1}^{q+r} a_{ij} f_{ij} = \sum_{i=1}^p \sum_{j=1}^q S_{ij}^{(r-1)} \Delta_{rr} f_{ij} + \sum_{i=1}^p \sum_{j=1}^r S_{i, q+j}^{(r-1)} \Delta_{r(r,i)} f_{i, q+j} \\ + \sum_{i=1}^r \sum_{j=1}^q S_{p+i, j}^{(r-1)} \Delta_{(r,i)} f_{p+i, j} + \sum_{i=1}^r \sum_{j=1}^r S_{p+i, q+j}^{(r-1)} \Delta_{(r,i)(r,j)} f_{p+i, q+j}.$$

The analogous identity for multiple series of order  $n$  is as follows:

$$(6) \quad \sum_{i_1=1}^{m_1+r} \cdots \sum_{i_n=1}^{m_n+r} a_{[i]} f_{[i]} = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} S_{[i]}^{(r-1)} \Delta_{r \cdots r} f_{[i]} \\ + \left[ \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^r S_{[i, 1, m]}^{(r-1)} \Delta_{r \cdots r(r, i_n)} f_{[i, 1, m]} \right] + \cdots \\ + \left[ \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-k}=1}^{m_{n-k}} \sum_{i_{n-k+1}=1}^r \cdots \sum_{i_n=1}^r S_{[i, k, m]}^{(r-1)} \Delta_{r \cdots r(r, i_{n-k+1}) \cdots (r, i_n)} f_{[i, k, m]} \right] \\ + \cdots + \left[ \sum_{i_1=1}^r \cdots \sum_{i_n=1}^r S_{[i, n, m]}^{(r-1)} \Delta_{(r, i_1) \cdots (r, i_n)} f_{[i, n, m]} \right],$$

where the small bracket symbol  $[i, k, m]$ , for  $k=1, 2, \dots, n$ , stands for the set of indices  $i_1, \dots, i_{n-k}, m_{n-k+1}+i_{n-k+1}, \dots, m_n+i_n$ , and the large brackets inclosing an expression indicate the sum of the whole group of terms of similar form, obtained by permutation of indices.

3. **Theorems on convergence factors of type I.** In this section we shall state the theorems of this nature for the case of double series and of multiple series of order  $n > 2$ .

**THEOREM I.** *The necessary and sufficient conditions that the double series  $\sum a_{ij} f_{ij}(\alpha, \beta)$  converge for all values of  $\alpha$  and  $\beta$  included in a set  $E(\alpha, \beta)$ , whenever the series  $\sum a_{ij}$  is summable  $(C, r-1)$  and satisfies the condition*

$$(7) \quad |S_{ij}^{(r-1)} / A_{ij}^{(r-1)}| < C \quad (i, j = 1, 2, \dots, C \text{ a constant})$$

*are that the convergence factors  $f_{ij}(\alpha, \beta)$  satisfy the following conditions:*

\* Loc. cit., pp. 88, 89.

† The use of the index  $(r-1)$  instead of  $k$  is for the purpose of simplifying the writing of the identity.

$$(A) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^{r-1} j^{r-1} |\Delta_{rj} f_{ij}(\alpha, \beta)| < K(\alpha, \beta) \quad (E(\alpha, \beta));$$

$$(B_1) \quad \lim_{j \rightarrow \infty} j^{r-1} \sum_{i=1}^p i^{r-1} |\Delta_{r0} f_{ij}(\alpha, \beta)| = 0 \quad (E(\alpha, \beta); p = 1, 2, \dots);$$

$$(B_2) \quad \lim_{i \rightarrow \infty} i^{r-1} \sum_{j=1}^q j^{r-1} |\Delta_{0r} f_{ij}(\alpha, \beta)| = 0 \quad (E(\alpha, \beta); q = 1, 2, \dots);$$

$$(C) \quad i^{r-1} j^{r-1} |f_{ij}(\alpha, \beta)| < M(\alpha, \beta) \quad (E(\alpha, \beta); i, j = 1, 2, \dots),$$

where  $K(\alpha, \beta)$  and  $M(\alpha, \beta)$  are finite for each pair of values of  $(\alpha, \beta)$  in  $E(\alpha, \beta)$ .

**THEOREM II.** *The necessary and sufficient conditions that the multiple series  $\sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n}(\alpha_1, \alpha_2, \dots, \alpha_n)$  converge for all values of  $(\alpha_1, \dots, \alpha_n)$  included in a set  $E(\alpha_1, \dots, \alpha_n)$ , whenever the series  $\sum a_{i_1, \dots, i_n}$  is summable  $(C, r-1)$  and satisfies the condition*

$$(8) \quad |S_{[i]}^{(r-1)} / A_{[i]}^{(r-1)}| < C \quad (i_1, \dots, i_n = 1, 2, \dots; C \text{ a constant}),$$

are that the convergence factors  $f_{[i]}(\alpha_1, \dots, \alpha_n)$  satisfy the following conditions:

$$(A) \quad \sum_{i_1=1}^{\infty} \dots \sum_{i_n=1}^{\infty} (i_1 \dots i_n)^{r-1} |\Delta_{r \dots r} f_{[i]}(\alpha_1, \dots, \alpha_n)| < K(\alpha_1, \dots, \alpha_n) \quad (E(\alpha_1, \dots, \alpha_n));$$

$n$  conditions  $(B_s^{(1)})$  of the type

$$(B_1^{(1)}) \quad \lim_{i_n \rightarrow \infty} i_n^{r-1} \sum_{i_1=1}^{p_1} \dots \sum_{i_{n-1}=1}^{p_{n-1}} (i_1 \dots i_{n-1})^{r-1} \Delta_{r \dots r 0} f_{[i]}(\alpha_1, \dots, \alpha_n) = 0 \quad (E(\alpha_1, \dots, \alpha_n); p_1, \dots, p_{n-1} = 1, 2, \dots);$$

$(n)$  conditions  $(B_s^{(k)})$  of the type

$$(B_1^{(k)}) \quad \lim_{(i_{n-k+1}, \dots, i_n \rightarrow \infty)} (i_{n-k+1} \dots i_n)^{r-1} \sum_{i_1=1}^{p_1} \dots \sum_{i_{n-k}=1}^{p_{n-k}} (i_1 \dots i_{n-k})^{r-1} \Delta_{r \dots r 0 \dots 0} f_{[i]}(\alpha_1, \dots, \alpha_n) = 0 \quad (E(\alpha_1, \dots, \alpha_n); p_1, \dots, p_{n-k} = 1, 2, \dots),$$

where the first  $n-k$  subscripts of  $\Delta$  have the value  $r$ , and the  $k$  remaining subscripts are equal to 0,  $k$  ranging from 2 to  $n-1$  inclusive;

$$(C) \quad |(i_1 \dots i_n)^{r-1} f_{[i]}(\alpha_1, \dots, \alpha_n)| < M(\alpha_1, \dots, \alpha_n) \quad (E(\alpha_1, \dots, \alpha_n); i_1, \dots, i_n = 1, 2, \dots);$$

where  $K(\alpha_1, \dots, \alpha_n)$  and  $M(\alpha_1, \dots, \alpha_n)$  are finite and positive for each set of values of  $(\alpha_1, \dots, \alpha_n)$  in  $E(\alpha_1, \dots, \alpha_n)$ .

4. **Sufficiency of the conditions.** We represent by  $S$  the value to which  $\sum a_{ij}$  is summable, and we introduce the series  $\sum \bar{a}_{ij}$ , where  $\bar{a}_{11} = a_{11} - S$  and the other  $\bar{a}$ 's are identical with the corresponding  $a$ 's. If we form  $\bar{S}_{ij}^{(r-1)}$  from the series  $\sum \bar{a}_{ij}$  in the same way that  $S_{ij}^{(r-1)}$  was formed from  $\sum a_{ij}$ , we have an identity of the form (5) between the  $\bar{a}_{ij}$  and the  $\bar{S}_{ij}^{(r-1)}$ . Since  $\bar{S}_{ij}^{(r-1)}/(ij)^{r-1}$  remains finite for all values of  $(i, j)$  and approaches zero as  $i$  and  $j$  become infinite, it follows from conditions (A), (B<sub>1</sub>), (B<sub>2</sub>), and (C) respectively that the first, second, third, and fourth terms\* on the right hand side of (5) approach limits as  $p$  and  $q$  become infinite. Hence the left hand side of (5) approaches a limit under the same conditions, and the sufficiency of our conditions is established.

If we allow  $p$  and  $q$  to become infinite in the identity of the form (5), we obtain the following identity:†

$$(9) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{a}_{ij} f_{ij}(\alpha, \beta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{S}_{ij} \Delta_{rr} f_{ij}(\alpha, \beta) \quad (E(\alpha, \beta)).$$

5. **Necessity of the conditions.** We consider first condition (C). If (C) were not satisfied for a certain choice of  $\alpha$  and  $\beta$ , we could find a set of values of  $i$  and  $j$ ,  $(p_1, q_1), (p_2, q_2), \dots, (p_n, q_n), \dots$ , such that  $p_{n+1} - p_n$  and  $q_{n+1} - q_n$  are greater than or equal to  $(r+1)$  for every  $n$ , and for which  $|(ij)^{r-1} f_{ij}(\alpha, \beta)| = M_{ij}$  becomes infinite as  $i$  and  $j$  become infinite by taking on these values successively. We then consider the series for which  $S_{ij}^{(r-1)} = (ij)^{r-1}/M_{ij}^{1/2}$  ( $i = p_1, p_2, \dots; j = q_1, q_2, \dots$ ) and  $S_{ij}^{(r-1)} = 0$  for all other values of  $i$  and  $j$ . For this series

$$a_{ij} = \Delta_{rr} S_{i-r, j-r}^{(r-1)} = (ij)^{r-1}/M_{ij}^{1/2} \quad (i = p_1, p_2, \dots; j = q_1, q_2, \dots).$$

Hence  $|a_{ij} f_{ij}| = M_{ij}^{1/2}$  ( $i = p_1, p_2, \dots; j = q_1, q_2, \dots$ ), and therefore  $a_{ij} f_{ij}$  does not approach a limit as  $i$  and  $j$  become infinite. Thus we have a contradiction, since the series  $\sum a_{ij} f_{ij}$  was supposed to be convergent.

We pass next to condition (A). If (A) does not hold for a certain choice of  $\alpha$  and  $\beta$ ,  $(\alpha_1, \beta_1)$ , we can select a set of values of  $p, p_1, p_2, \dots, p_n, \dots$ , such that if we define

$$\sigma_p = \sum_{i=1}^p \sum_{j=1}^p (ij)^{r-1} |\Delta_{rr} f_{ij}(\alpha_1, \beta_1)|,$$

\*In the case of multiple series of order  $n > 2$  the additional terms on the right hand side of the identity (6) are taken care of by means of the additional conditions of the form  $B_s^{(k)}$ .

†It should be noted that an identity of this form holds for any series  $\sum a_{ij}$  summable (C,  $r-1$ ) to zero, if the factors  $f_{ij}$  satisfy conditions (A), (B<sub>1</sub>), (B<sub>2</sub>), and (C).

we shall have

$$\sigma_{p_{n+1}} - \sigma_{p_{n+r}} > 1 \quad (n = 1, 2, \dots).$$

We may then obtain a contradiction from the identity (5) by using the special series for which

$$S_{ij}^{(r-1)} = [\text{sgn } \Delta_{rr} f_{ij}(\alpha_1, \beta_1)] \frac{(ij)^{r-1}}{n}$$

$$(p_n + r < i_j \leq p_{n+1}; 1 \leq j \leq p_{n+1}; n = 1, 2, \dots),$$

$S_{ij}^{(r-1)} = 0$  for all other values of  $(i, j)$ . For this series is summable  $(C, r-1)$  to the value zero and satisfies condition (7). Therefore  $\sum a_{ij} f_{ij}(\alpha_1, \beta_1)$  is convergent, or the left hand side of (5) approaches a limit for  $\alpha = \alpha_1, \beta = \beta_1$ , as  $p$  and  $q$  become infinite. But it is readily seen that with the above choice of  $S_{ij}^{(r-1)}$  the right hand side of (5) becomes infinite when  $p$  and  $q$  become infinite by taking on the special set of values  $(p_2, q_2), (p_3, q_3), \dots, (p_n, q_n), \dots$ . From this contradiction the necessity of (A) follows at once.

Since the proofs for the necessity of  $(B_1)$  and  $(B_2)$  are entirely analogous, we shall deal only with  $(B_1)$ . In the case of multiple series of order  $n > 2$ , the proof of the necessity of each of the  $(2^n - 2)$  conditions  $(B_s^{(k)})$  is analogous to the proof here given for condition  $(B_1)$  of Theorem I.

If  $(B_1)$  does not hold for a certain choice of  $\alpha$  and  $\beta, (\alpha_1, \beta_1)$ , we can find an  $\epsilon > 0$  and an  $m$  such that

$$q^{r-1} \sum_{i=1}^m i^{r-1} |\Delta_{r0} f_{iq}(\alpha_1, \beta_1)| > \epsilon$$

for an infinite number of choices of  $q, q_1, q_2, \dots, q_n, \dots$ , each one of which exceeds the preceding one by at least  $r$ .

We then consider the series for which

$$S_{iq_n}^{(r-1)} = (-1)^n (iq_n)^{r-1} [\text{sgn } \Delta_{r0} f_{iq_n}(\alpha_1, \beta_1)] \quad (1 \leq i \leq m, n = 1, 2, \dots),$$

$$S_{ij}^{(r-1)} = 0 \quad (\text{all other } i, j).$$

This series is summable  $(C, r-1)$  to the value zero and satisfies condition (7). If we allow  $p$  and  $q$  to become infinite in such a manner that  $q$  takes on the successive values  $q_n - r$  ( $n = 1, 2, \dots$ ), the second term on the right hand side of (5) will oscillate between values  $> \epsilon$  and  $< -\epsilon$ . The other terms on the right hand side, and the left hand side, will approach definite limits. Thus we shall have a contradiction, and the necessity of  $(B_1)$  is established.

Since, as we have already pointed out, the necessity of  $(B_2)$  may be proved in analogous fashion, it follows that all four of the conditions  $(A)$ ,  $(B_1)$ ,  $(B_2)$ , and  $(C)$  are necessary in order that every series summable  $(C, r-1)$  and satisfying condition (7) may be reduced to convergence by the introduction of the convergence factors  $f_{ij}(\alpha, \beta)$ .

6. **Theorems on convergence factors of type II.** We will now state the two theorems of this kind for the case of double series and of multiple series of order  $n > 2$ .

**THEOREM III.** *The necessary and sufficient conditions that the double series  $\sum a_{ij} f_{ij}(\alpha, \beta)$  should converge in  $E(\alpha, \beta)$  and should approach  $S$  as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$ , a limit point of  $E(\alpha, \beta)$  not included in that set, whenever the series  $\sum a_{ij}$  is summable  $(C, r-1)$  to the value  $S$  and satisfies condition (7), are that the convergence factors  $f_{ij}(\alpha, \beta)$  satisfy the conditions of Theorem I and the following further conditions:*

$$(A') \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^{r-1} j^{r-1} |\Delta_{rr} f_{ij}(\alpha, \beta)| < K \quad (E'(\alpha, \beta)),$$

$$(D_1) \quad \lim_{(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)} \sum_{j=q}^{\infty} j^{r-1} |\Delta_{rr} f_{ij}(\alpha, \beta)| = 0 \quad (i, q = 1, 2, \dots),$$

$$(D_2) \quad \lim_{(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)} \sum_{i=p}^{\infty} i^{r-1} |\Delta_{rr} f_{ij}(\alpha, \beta)| = 0 \quad (p, j = 1, 2, \dots),$$

$$(E) \quad \lim_{(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)} f_{ij}(\alpha, \beta) = 1 \quad (i, j = 1, 2, \dots),$$

where  $K$  is a positive constant and  $E'(\alpha, \beta)$  includes all points of  $E(\alpha, \beta)$  lying in a certain neighborhood of  $(\alpha_0, \beta_0)$ .

**THEOREM IV.** *The necessary and sufficient conditions that the multiple series  $\sum a_{[i]} f_{[i]}(\alpha_1, \alpha_2, \dots, \alpha_n)$  should converge in  $E(\alpha_1, \dots, \alpha_n)$  and should approach  $S$  as  $(\alpha_1, \dots, \alpha_n) \rightarrow (\alpha_1^{(0)}, \dots, \alpha_n^{(0)})$ , a limit point of  $E(\alpha_1, \dots, \alpha_n)$  not included in that set, whenever the series  $\sum a_{[i]}$  is summable  $(C, r-1)$  to the value  $S$  and satisfies condition (8), are that the convergence factors  $f_{[i]}(\alpha_1, \dots, \alpha_n)$  satisfy the conditions of Theorem II and the following further conditions:*

$$(A') \quad \sum_{i_1=1}^{\infty} \dots \sum_{i_n=1}^{\infty} (i_1 \dots i_n)^{r-1} |\Delta_{r \dots r} f_{[i]}(\alpha_1, \dots, \alpha_n)| < K \quad (E'(\alpha_1, \dots, \alpha_n)),$$

$n$  conditions  $(D_s)$  of the type

$$(D_1) \quad \lim_{(\alpha_1, \dots, \alpha_n) \rightarrow (\alpha_1^{(0)}, \dots, \alpha_n^{(0)})} \sum_{i_2=q_2}^{\infty} \cdots \sum_{i_n=q_n}^{\infty} (i_2 \cdots i_n)^{r-1} |\Delta_{r \dots r} f_{[i]}(\alpha_1, \dots, \alpha_n)| = 0$$

$$(i_1, q_2, \dots, q_n = 1, 2, \dots),$$

and

$$(E) \quad \lim_{(\alpha_1, \dots, \alpha_n) \rightarrow (\alpha_1^{(0)}, \dots, \alpha_n^{(0)})} f_{[i]}(\alpha_1, \dots, \alpha_n) = 1 \quad (i_1, \dots, i_n = 1, 2, \dots),$$

where  $K$  is a positive constant and  $E'(\alpha_1, \dots, \alpha_n)$  includes all points of  $E(\alpha_1, \dots, \alpha_n)$  lying in a certain neighborhood of  $(\alpha_1^{(0)}, \dots, \alpha_n^{(0)})$ .

7. **Sufficiency of the conditions.** If we introduce the notation  $\bar{a}_{ij}$  and  $\bar{S}_{ij}$  with the same significance as in the proof of Theorem I, the identity (9) follows from conditions  $(A)$ ,  $(B_1)$ ,  $(B_2)$ , and  $(C)$ . We will show that under conditions  $(A')$ ,  $(D_1)$ ,  $(D_2)$ , and  $(E)$  the right hand side of (9) approaches zero as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$ . It will follow that the left hand side of (9) approaches zero under similar circumstances, and that therefore  $\sum a_{ij} f_{ij}(\alpha, \beta)$  approaches  $S$  as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$ . Thus the sufficiency of our conditions will be established. In the case of multiple series of order  $n > 2$  the proof follows similar lines, conditions  $(D_s)$  ( $s = 1, 2, \dots, n$ ), being used in a manner analogous to the use of conditions  $(D_1)$  and  $(D_2)$ .

Given an arbitrary positive  $\epsilon$ , we choose  $(p, q)$  so that

$$[\bar{S}_{ij}^{(r-1)} / (ij)^{r-1}] < (\epsilon / 4K) \quad (i \geq p, j \geq q).$$

It then follows from  $(A')$  that

$$(10) \quad \left| \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} S_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \right| < \frac{\epsilon}{4} \quad (E'(\alpha, \beta)).$$

In view of  $(D_1)$  and  $(D_2)$  and condition (7), it follows that there is a certain neighborhood of  $(\alpha_0, \beta_0)$  such that for the set  $E''(\alpha, \beta)$  of  $E(\alpha, \beta)$ , lying in this neighborhood,

$$(11) \quad \left| \sum_{i=1}^{p-1} \sum_{j=q}^{\infty} S_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \right| < \frac{\epsilon}{4}$$

$$\left| \sum_{i=p}^{\infty} \sum_{j=1}^{q-1} S_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \right| < \frac{\epsilon}{4} \quad (E''(\alpha, \beta)).$$

From (E) it follows that a set  $E'''(\alpha, \beta)$ , lying in  $E(\alpha, \beta)$ , may be found such that

$$(12) \quad \left| \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} S_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \right| < \frac{\epsilon}{4} \quad (E'''(\alpha, \beta)).$$

If we represent by  $\bar{E}(\alpha, \beta)$  the set of points that is common to  $E'$ ,  $E''$ , and  $E'''$ , we have from (10), (11), and (12),

$$\left| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \bar{S}_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \right| < \epsilon \quad (\bar{E}(\alpha, \beta)).$$

Hence  $\sum \bar{S}_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta) \rightarrow 0$  as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$  over values included in  $E(\alpha, \beta)$ , and as pointed out above, the sufficiency of our conditions is established.

8. **Necessity of the conditions.** We consider first condition (E). Choose a double series  $\sum a_{ij}$  which is summable  $(C, r-1)$  to a value  $S \neq 0$ , and represent by  $\sum \bar{a}_{ij}$  the series for which  $\bar{a}_{mn} = a_{mn} - S$ ,  $a_{mn}$  being any term of  $\sum a_{ij}$ , and the other terms  $\bar{a}_{ij}$  are identical with the terms  $a_{ij}$ . It then follows that the series  $\sum \bar{a}_{ij}$  is summable  $(C, r-1)$  to zero, and hence by our hypothesis the series  $\sum a_{ij} f_{ij}(\alpha, \beta)$  and  $\sum \bar{a}_{ij} f_{ij}(\alpha, \beta)$  are each convergent in  $E(\alpha, \beta)$ . We have obviously

$$\sum \bar{a}_{ij} f_{ij}(\alpha, \beta) + S f_{mn}(\alpha, \beta) = \sum a_{ij} f_{ij}(\alpha, \beta) \quad (E(\alpha, \beta)).$$

If we let  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$ , it follows from our hypothesis that the first term on the left hand side of this equation approaches zero and that the right hand side approaches  $S$ . Hence the second term on the left hand side approaches  $S$  as a limit as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$ , and consequently

$$\lim_{(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)} f_{mn}(\alpha, \beta) = 1 \quad (m, n = 1, 2, \dots).$$

Thus the necessity of (E) is established.

Consider next (A'). For a double series which is summable  $(C, r-1)$  to zero, we have the identity

$$F(\alpha, \beta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} f_{ij}(\alpha, \beta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} S_{ij}^{(r-1)} \Delta_{rr} f_{ij}(\alpha, \beta),$$

whenever the series in the second member converges in  $E(\alpha, \beta)$  and condition (7) is satisfied. For by Theorem I conditions (A),  $(B_1)$ ,  $(B_2)$ , and (C) are necessarily satisfied under the hypotheses stated, and therefore the above identity holds.\* If (A') is not satisfied for a certain set  $E'(\alpha, \beta)$ ,

\*Cf. the second footnote, §4.

having  $(\alpha_0, \beta_0)$  as a limit point, we can select from  $E(\alpha, \beta)$  a sequence of  $(\alpha, \beta)$ 's,  $(\alpha^{(m)}, \beta^{(m)})$ , approaching  $(\alpha_0, \beta_0)$  and such that

$$(13) \quad \sum_{i=1}^p \sum_{j=1}^p (ij)^{r-1} |\Delta_{rrf_{ij}}(\alpha^{(m)}, \beta^{(m)})| > m \quad (m = 1, 2, \dots)$$

for a proper choice of  $p$ . Choose  $m_1 \geq 3$ , and let  $(\alpha_1, \beta_1)$  and  $p_1$  be the corresponding values of  $(\alpha, \beta)$  and  $p$ . Take  $s_1 > p_1$ , and define

$$S_{ij}^{(r-1)} = [\operatorname{sgn} \Delta_{rrf_{ij}}(\alpha_1, \beta_1)] (ij)^{r-1} \quad (1 \leq \frac{i}{j} \leq s_1).$$

Now choose  $(\alpha_2, \beta_2)$  from the sequence  $(\alpha^{(m)}, \beta^{(m)})$  such that\*

$$\sum_{i=1}^{s_1} \sum_{j=1}^{s_1} (ij)^{r-1} |\Delta_{rrf_{ij}}(\alpha_2, \beta_2)| < \frac{1}{4 \log m_1},$$

and let  $m_2$  and  $p_2$  be the corresponding values of  $m$  and  $p$  for which the inequality (13) holds. Take  $s_2 > p_2$  and such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (ij)^{r-1} |\Delta_{rrf_{ij}}(\alpha_2, \beta_2)| - \sum_{i=1}^{s_1} \sum_{j=1}^{s_1} (ij)^{r-1} |\Delta_{rrf_{ij}}(\alpha_2, \beta_2)| < \frac{1}{4 \log m_1}.$$

We then define

$$S_{ij}^{(r-1)} = [\operatorname{sgn} \Delta_{rrf_{ij}}(\alpha_2, \beta_2)] \frac{(ij)^{r-1}}{\log m_1} \quad \left( \begin{array}{l} s_1 < i \leq s_2, \quad 1 \leq j \leq s_2 \\ 1 \leq i \leq s_1, \quad s_1 < j \leq s_2 \end{array} \right).$$

Continuing in this fashion, we define a series which is summable  $(C, r-1)$  to zero, while at the same time  $|F(\alpha_n, \beta_n)| > (m_n/4 \log m_{n-1})$  ( $n=2, 3, \dots$ ), so that  $F(\alpha, \beta)$  does not tend to a limit as  $(\alpha, \beta) \rightarrow (\alpha_0, \beta_0)$  over any set in  $E(\alpha, \beta)$ . Thus we have a contradiction and the necessity of  $(A')$  is established.

The proofs of the necessity of  $(D_1)$  and  $(D_2)$  are entirely analogous, and we will therefore consider only the case of  $(D_1)$ . In the case of multiple series of order  $n > 2$  the proof for each of the various conditions  $(D_s)$  is analogous to the proof for  $(D_1)$  here given.

If  $(D_1)$  does not hold, we can find an  $\epsilon$ , an  $m$ , and a  $q$  such that

$$\sum_{j=q}^{\infty} j^{r-1} |\Delta_{rrf_{mj}}(\alpha, \beta)|$$

\*That such an  $(\alpha_2, \beta_2)$  exists follows from the necessity of  $(E)$ , already established.

exceeds  $\epsilon$  for an infinite set of values of  $(\alpha, \beta)$  lying in  $E(\alpha, \beta)$  and having  $(\alpha_0, \beta_0)$  as a limit point. Choose one pair of values  $(\alpha_1, \beta_1)$  from this sequence and determine  $s_1$  such that

$$\sum_{j=q}^{s_1} j^{r-1} |\Delta_{rrf_{mj}}(\alpha_1, \beta_1)| > \frac{3\epsilon}{4}, \quad \sum_{j=s_1+1}^{\infty} j^{r-1} |\Delta_{rrf_{mj}}(\alpha_1, \beta_1)| < \frac{\epsilon}{4}.$$

Then take

$$S_{mj}^{(r-1)} = [\operatorname{sgn} \Delta_{rrf_{mj}}(\alpha_1, \beta_1)] j^{r-1} \quad (q \leq j \leq s_1).$$

Now find  $(\alpha_2, \beta_2)$  such that

$$\sum_{j=q}^{s_1} j^{r-1} |\Delta_{rrf_{mj}}(\alpha_2, \beta_2)| < \frac{\epsilon}{8}$$

and choose  $s_2$  so that

$$\sum_{j=s_1+1}^{s_2} j^{r-1} |\Delta_{rrf_{mj}}(\alpha_2, \beta_2)| > \frac{3\epsilon}{4}, \quad \sum_{j=s_2+1}^{\infty} j^{r-1} |\Delta_{rrf_{mj}}(\alpha_2, \beta_2)| < \frac{\epsilon}{8}.$$

Then take

$$S_{mj}^{(r-1)} = [\operatorname{sgn} \Delta_{rrf_{mj}}(\alpha_2, \beta_2)] j^{r-1} \quad (s_1 < j \leq s_2).$$

Continue this process for the choice of  $S_{mj}^{(r-1)} (j \geq q)$ , and choose all other  $S_{ij}^{(r-1)} = 0$ . Thus we shall obtain a series which is summable  $(C, r-1)$  to zero, whereas  $F(\alpha_n, \beta_n) > \frac{1}{2}\epsilon$  ( $n=1, 2, \dots$ ). This contradiction of our hypothesis establishes the necessity of  $(D_1)$ , and as stated above the necessity of  $(D_2)$  can be proved in a manner entirely analogous.

UNIVERSITY OF CINCINNATI,  
CINCINNATI, OHIO

